

# Cheshire Cat resurgence, Self-resurgence and Quasi-Exact Solvable Systems

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**ABSTRACT:** We explore a one parameter  $\zeta$ -deformation of the quantum-mechanical Sine-Gordon and Double-Well potentials which we call the Double Sine-Gordon (DSG) and the Tilted Double Well (TDW), respectively. In these systems, for positive integer values of  $\zeta$ , the lowest  $\zeta$  states turn out to be exactly solvable for DSG – a feature known as Quasi-Exact-Solvability (QES) – and solvable to all orders in perturbation theory for TDW. For DSG such states do not show any instanton-like dependence on the coupling constant, although the action has real saddles. On the other hand, although it has no real saddles, the TDW admits all-orders perturbative states that are not normalizable, and hence, requires a non-perturbative energy shift. Both of these puzzles are solved by including complex saddles. We show that the convergence is dictated by the quantization of the hidden topological angle. Further, we argue that the QES systems can be linked to the exact cancellation of real and complex non-perturbative saddles to all orders in the semi-classical expansion. Further, we show that the entire resurgence structure remains encoded in the analytic properties of the  $\zeta$ -deformation, even though exactly at integer values of  $\zeta$  the mechanism of resurgence is obscured by the lack of ambiguity in both the Borel sum of the perturbation theory as well as the non-perturbative contributions. In this way, all of the characteristics of resurgence remains even when its role seems to vanish, much like the lingering grin of the mythological Cheshire Cat. We also show that the perturbative series is Self-resurgent – a feature by which there is a one-to-one relation between the early terms of the perturbative expansion and the late terms of *the same expansion* – which is intimately connected with the Dunne-Ünsal relation. We explicitly check verify that this is indeed the case.

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# 1 Introduction and Results

For many quantum systems, perturbation theory is widely employed successfully to obtain approximate results. Following the immense success of the perturbative treatment of Quantum Electrodynamics which resulted in a Nobel Prize shared by Tomonaga, Schwinger and Feynman in 1965, the perturbation theory and Feynman diagrams became firmly associated with *the Quantum Field Theory*. Indeed, concepts such as renormalization group, Bjorken scaling, running coupling and asymptotic freedom are just some of the concepts intimately tied to the utility and indispensability of the perturbation theory.

Nevertheless, already as early as 1952, Dyson gave a physical argument that analytic continuation of the electric charge  $e^2 \rightarrow -e^2$  would cause an instability, effectively indicating that in Quantum Electrodynamics — the simplest and most accurately verified quantum field theory manifested in nature — the radius of convergence of the perturbation theory is zero [1]. Since then it has become clear that this is a generic feature of both quantum mechanical as well as field theoretical systems, with a typical divergence rate being factorial. It is for this reason that perturbation theory fails to define a quantum field theory, or even, indeed, quantum mechanics.

Perhaps unsurprisingly while the successes of the perturbation theory are commonly praised and a matter of textbook knowledge, its apparent deficits deeply rooted in its structure are more often than not either overlooked or tacitly ignored. While this point of view is sometimes necessary and sometimes useful, it turns a blind eye to the beautiful and intricate structure hidden in the perturbative expansion and its stubborn insistence on diverging. The question then what, if any, is the meaning of such series.

A celebrated way to make sense of factorially diverging series is to tame them by a special transformation — the Borel transform, which renders the series convergent. The Laplace transform of this sum gives rise to another function — the Borel sum — which has the same asymptotic expansion as the original series but assigns a non-divergent value to it. If this can be done in a unique way, the series is said to be *Borel summable*, as there is a sense in which a divergent series is assigned a concrete value. Still the Borel transform often has singularities in generic cases which may render the Borel sum ambiguous. The goal of resurgence theory is to describe the global nature of the solution by analyzing these singularities and ambiguities that they may cause [2–6]. For instance, if those singularities lie on the positive real axis, we might have to avoid these poles by going around them in the complex plane. Different deformations introduce imaginary ambiguities for physical observables, e.g. for energy levels. At first, we might be tempted to abandon this prescription due to this kind of pathological results once and for all. Nevertheless remarkably and perhaps surprisingly the pathology of the perturbation theory turns out to be inextricably linked to the non-perturbative physics [7–16]. In other words, the ambiguity caused by the divergence and non-Borel summability of the small coupling expansion serves as a placeholder, much like a pattern of a jigsaw puzzle, stitching perturbative and non-perturbative contributions in such a way to eliminate all ambiguities. The study, analysis and understanding of such phenomena are known under the name of *resurgence theory*.

The resurgence theory developed by Écalle [2] is proficient enough to encode the subtle information around different saddles by replacing the conventional perturbation series with transseries. The transseries do not only consist of a power series in the coupling constant but also include non-analytic terms relevant to instanton contributions and the integration over their quasi-moduli. The power of the resurgence theory lies in the possibility that it may provide a consistent manner to take into account the presence of all saddle points under certain physical requirements. More concretely, it connects the perturbative fluctuations around different saddles via intricate relations with each other

and respects the monodromy properties of the underlying quantum system. There has been an ever growing set of physical systems where resurgence theory resolves some puzzles and reveals surprises related to semi-classical analysis, such as the semi-classical interpretation and the role of renormalon-like singularities [17–22], the relation between perturbation theories between different saddles [9, 23–26], stabilization of center symmetry in super Yang-Mills theory [27, 28], Borel summability of  $\mathcal{N} = 2$  super Yang-Mills [17, 18, 29], the meaning and limits of the Bogomolny–Zinn-Justin prescription [30, 31], the vanishing gluon condensate in SUSY gauge theories [30], the role of multi-instantons in  $\mathcal{N} = 1$  [30, 32] and  $\mathcal{N} = 2$  [31] quantum mechanics, the meaning and limits of the Bogomolny–Zinn-Justin prescription [31], role of “instantons” and complex solutions in the Gross-Witten theory [33], as well as an abundance of work ranging from quantum mechanics to general quantum field theory to string theory [10, 14, 16, 34–43].

In this work, we aim to solve yet another puzzle related to systems for which a part of the spectrum can be solved at isolated points in the parameter space. Such special systems are dubbed *Quasi-Exactly Solvable (QES) systems* pioneered by Turbiner [44, 45, 45, 46], and as a rule they never have essential singularities of the type  $e^{-1/g}$ . Further, the perturbation theory in QES for the relevant part of the spectrum systems is convergent. However, such system often have real non-trivial saddles for which it seems impossible to argue that they do not contribute, in contradiction with the absence of contributions of the type  $e^{-1/g}$ . Further, related set of systems which we call pseudo-QES systems have a completely convergent perturbation theory even though they cannot be solved for exactly.

In both QES and pseudo-QES systems we could rightfully argue that there is no need of Borel sum since the perturbation series is convergent and therefore well-defined. A trivial example of this situation is already given by the ground state of the SUSY Quantum Mechanics [47], which is zero to all orders of perturbation theory. Because of this the Borel plane is free from singularities, rendering the perturbation theory trivially unambiguous. Hence, one may be tempted to conclude that the perturbative and non-perturbative effects are completely disconnected. While this statement is not in contradiction with resurgence, it would appear that the role of resurgence in these systems is trivial as the different sectors appear to be independent from each other and no cancellation among them is required. We are going to argue that in reality the situation is more subtle, and that resurgence is still governing the interplay between different sectors encoded in the analytical properties of all the various contributions. So, much like the grin of the mythological Cheshire cat, resurgent properties linger even when its main role seems to vanish. For this reason we call this property the Cheshire Cat resurgence<sup>1</sup>.

Finally we discuss a remarkable property of the perturbative expansion of the energy levels in these systems: the self-resurgence. Namely because the crucial contributions to the energy is coming from a complex saddle, which generically gives a complex contribution to the energy (except when the hidden topological angle is quantized). By requiring the resurgence cancellation to take place, the complex saddle and the perturbative expansion around it has a one-to-one relation with the large order growth of the perturbation theory. On the other hand the early terms of perturbative corrections *around the complex saddle solution* can be directly connected to the early terms of the perturbative expansion *around the trivial vacuum* via the generalization of Dunne–Ünsal relation [10] to these system. By transitivity we are therefore able establish a one-to-one relation between the early terms of the perturbation theory and the late-asymptotic terms *of the same series*.

It is possible that this remarkable property of the perturbation series is connected to the work of

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<sup>1</sup>Many thanks to Thomas Schäfer pointing out the analogy.

Dingle<sup>2</sup> [48] where self-resurgence appears in expansions of functions which are themselves resurgence functions.

In this work, we address the following questions to shed more light on the resurgent structure of the perturbation theory and to better understand the relevance of the resurgence theory in quantum systems:

- *Does an all-order convergent perturbation theory give an exact answer?*
- *What is the role (if any) of the non-perturbative saddles?*

We explore the connections between various approaches to quantum mechanical systems with the goal to generalize them to quantum field theory:

- 1) The nature of perturbation theory: convergent *vs.* asymptotic,
- 2) The nature of complex saddles: quantized *vs.* unquantized hidden topological angles (associated with the saddles of holomorphized path integrals),
- 3) Supersymmetry and QES *vs.* non-solvability,<sup>3</sup>
- 4) Resurgence in disguise *vs.* explicit resurgence.

We demonstrate that these properties are intimately related: the left and right of the *vs.* in four categories are interconnected.

We study a one-parameter,  $\zeta$ , family of quantum mechanical systems. Varying  $\zeta$  will allow us to interpolate between a purely bosonic theory and quantum mechanical systems with a number of fermions. The integer values of  $\zeta$  are particularly interesting since we recover the simplest supersymmetric quantum mechanics when  $\zeta = 1$ , and for other positive integer values of  $\zeta$  the lowest  $\zeta$  eigenstates are algebraically solvable. As soon as  $\zeta$  differs from an integer value, the system ceases to be solvable, and its perturbation series become divergent. For this one-parameter family of quantum systems, the ones with analytic perturbation series consist of a measure-zero subset, and live as limits of generic values of  $\zeta$ . In other words, the resurgence theory connects the perturbative and non-perturbative sectors and guarantees the well-definiteness of the system for any generic value of  $\zeta$ . All these relations survive the special values of  $\zeta$  as well, which is a Cheshire Cat resurgence.

We unify various approaches to understand quantum mechanical systems parametrized by  $\zeta$ : Studies of the perturbation theory via Bender-Wu method<sup>4</sup> [49–51], the semi-classical analysis and holomorphization of path integral within Picard-Lefschetz theory [15, 30], supersymmetric quantum mechanics [52, 53] and quasi-exact solvability [54, 55], and resurgence theory applied to quantum mechanics [9, 10, 56]. In the course of exploring these connection, we also resolve some old standing puzzles in the literature of these topics mentioned above. For various known things, we give new streamlined arguments. In the remainder of the Introduction we will introduce the two models we study and review the main conclusions of the paper.

The paper is organized as follows: In the rest of this section we discuss our setup, our main results and the two puzzles related to the QES systems. Sections 2 and 3 are dedicated to the detailed resolution of the two puzzles, the Cheshire Cat resurgence and the self-resurgence properties in DSG and TDW systems, respectively. In 4 we discuss possible connections and parallels with QFT, while in 5 we give conclusions and summary.

<sup>2</sup>We are thankful to M.V. Berry for sharing the early manuscript with us.

<sup>3</sup>Algebraically non-solvable systems may potentially be solvable in the sense of a resurgent-transseries.

<sup>4</sup>The mathematica package `BenderWu` developed in [49] was used throughout this work. An up to date version can be freely obtained from the Wolfram package repository at <http://library.wolfram.com/infocenter/MathSource/9479/>.

### 1.1 The fermions and the $\zeta$ -deformed systems

The main outcome of this work is most simply described by considering the Euclidean bosonic Lagrangians of the type,

$$\mathcal{L}_\zeta^E = \frac{1}{g} \left( \frac{1}{2} \dot{x}^2 + V(x) \right) \quad V(x) = \frac{1}{2} (W'(x))^2 + \frac{1}{2} \zeta g W''(x), \quad (1.1)$$

where  $W(x)$  is auxiliary potential,  $V(x)$  is the potential,  $g$  is coupling, and  $\zeta$  is a deformation parameter whose consequence we explore. We say that the theory is purely bosonic when  $\zeta = 0$ . The  $\zeta = \pm 1$  cases are Fermi-Bose sectors (or spin up/down sectors) of supersymmetric quantum mechanics, where  $W(x)$  is called the super-potential. We further assume that there are instanton solutions in the purely bosonic theory, but this assumption can be dropped for generalization.

Quantum mechanics defined by the Lagrangian (1.2) has a formal similarity with some quantum field theories, such as adjoint QCD. To give enough motivation, let us consider a quantum mechanical systems with one bosonic and  $N_f$  Grassmann valued fields:

$$\mathcal{L}^E = \frac{1}{g} \left( \frac{1}{2} \dot{x}^2 + \frac{1}{2} (W'(x))^2 \right) + \frac{1}{2} i (\bar{\psi}_i \dot{\psi}_i - \dot{\bar{\psi}}_i \psi_i) + \frac{1}{2} W''(x) [\bar{\psi}_i, \psi_i], \quad i = 1, \dots, N_f. \quad (1.2)$$

Because of the fermion flavor symmetry of (1.2), this quantum system is decomposed into superselection sectors defined by fermion number  $k$  with degeneracy  $\binom{N_f}{k}$ . The Hamiltonian for the level  $k$  is

$$\hat{H}_{(N_f, k)} = \frac{g}{2} \frac{\hat{p}^2}{2} + \frac{1}{2g} (W'(x))^2 + \frac{1}{2} (2k - N_f) W''(x), \quad k = 0, \dots, N_f. \quad (1.3)$$

We now find that  $\zeta$  in (1.2) is a generalization of  $2k - N_f$ , and the  $\frac{1}{2} \zeta g W''(x)$  term should be viewed as a fermion loop effect.

The Lagrangian (1.2) is inspired from multi-flavor quantum field theory studies. For example, consider a non-linear sigma model in 2d and add to it a fermionic super-partner. And then, continue adding  $N_f$  fermionic flavors [19, 20]. Or similarly, consider adding adjoint representation fermions to 4d Yang-Mills, which becomes supersymmetric at  $N_f = 1$  and some multi-flavor theory for  $N_f \geq 2$ . There is by now building up evidence that these QFTs are special in some ways, and carry over some of the interesting aspects of supersymmetric theory [57, 58].

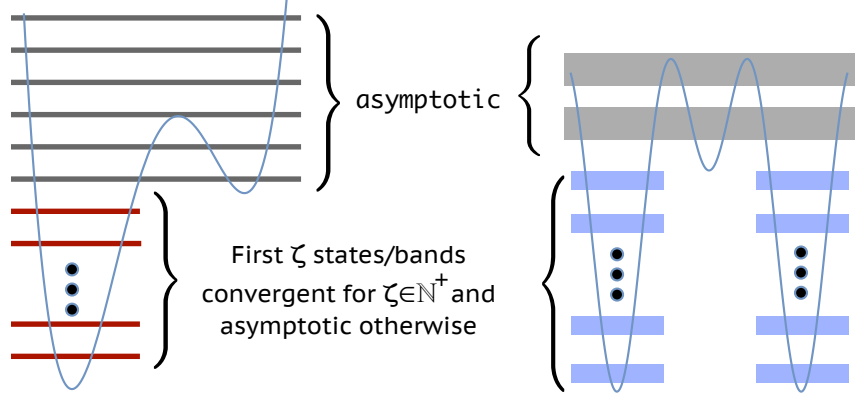
The QM systems with Lagrangians (1.2) also appear in other contexts. For example,

- Bosonic coordinate  $x(t)$  and one Grassmann valued coordinate  $\psi(t)$ , with a deformation of the Yukawa term by the parameter  $\zeta$ .
- Bosonic coordinate  $x(t)$ , and  $W''(x(t))$  coupled to spin in Bloch representation, via an abelian Berry phase term.  $\zeta$  acquires an interpretation as analytic continuation of spin quantum number.

See Ref. [15] for a more detailed discussion.

### 1.2 The nature of the perturbation theory

In general, the perturbation theory in powers of coupling constant  $g$  is a divergent asymptotic expansion because of the factorial growth of coefficients. According to resurgence, the asymptotic nature of the perturbation series is caused by the existence of the other saddles of the action (see, e.g., [11–13] for examples of one-dimensional integrals). A way to make sense out of divergent asymptotic expansion is the lateral Borel sum, i.e, directional Laplace integration of the Borel transform. Borel sum assigns a



**Figure 1.** For integer  $\zeta$ , perturbation theory for lowest  $\zeta$  states is convergent (e.g.  $\zeta = 1$  is the supersymmetric case), but others are divergent. For non-integer  $\zeta$ , perturbation theory for all states is asymptotic. The energy bands of the DSG system describe dependence of energy levels on the topological  $\theta$  angle.

holomorphic function on a Stokes sector to the asymptotic series. At least, in one-dimensional integrals, the geometric realization of Borel resummation is the integration over the Lefschetz thimbles, see for example [14]. This procedure identifies late terms of the perturbation series with early terms of the asymptotic expansion around another saddle.

In the case of quantum mechanics, the Planck constant  $\hbar$  or the coupling  $g$  is an expansion parameter and takes positive values. In many examples, the Borel transform has a singularity on the positive real axis, which causes an imaginary ambiguity in the Borel sum. Bogomolny and Zinn-Justin illustrated that the ambiguity is canceled by an instanton–anti-instanton contribution [7, 8], and this motivates that resurgence works also for quantum mechanics. This was originally demonstrated for the leading asymptotic growth, but its generalization to all orders is given in [9, 10]. There is a proposal to give a geometric interpretation of this Bogomolny–Zinn-Justin prescription in terms of Lefschetz thimbles, see the discussion in Section 4 of [15] and [16].

In order to tell our story more concretely, we will use two exemplary Hamiltonian

$$H = \frac{1}{2} [p^2 + W'(x)^2 + \zeta W''(x)], \quad (1.4)$$

with

$$\begin{aligned} W(x) &= -\omega \cos x, & \text{Double Sine Gordon (DSG)} \\ W(x) &= \frac{x^3}{3} - \frac{\omega^2}{4}x, & \text{Tilted Double Well (TDW)} \end{aligned} \quad (1.5)$$

for general values of  $\zeta$ . However, we believe our findings generalize to all potentials discussed [55] straightforwardly, as well as to all of the Quasi-Exactly solvable systems [44, 45, 45, 46].

Let  $E^{\text{pert.}}(\nu, g, \zeta)$  denote perturbative expansion of the energy for the level number  $\nu$  ( $\nu = 0$  is the ground state)

$$E^{\text{pert.}}(\nu, g, \zeta) = \sum_{n=0}^{\infty} a_n(\nu, \zeta) g^n = a_0(\nu, \zeta) + a_1(\nu, \zeta) g + a_2(\nu, \zeta) g^2 + \cdots, \quad (1.6)$$

By examining the large-orders of perturbation theory by using the method of Bender and Wu [50, 51] generalized in [49] to arbitrary potentials, we find that the large-order behavior of the expansion coefficients for a level  $\nu$  behaves as<sup>5</sup>

$$a_n(\nu, \zeta) \sim -\frac{\mathcal{M}}{2\pi} \frac{1}{\nu!} \frac{1}{(2A^2)^{\zeta-2\nu-1}} \frac{1}{\Gamma(1+\nu-\zeta)} \frac{(n-\zeta+2\nu)!}{(S_b)^{n-\zeta+2\nu+1}} \times \left( b_0(\nu, \zeta) + \frac{S_b b_1(\nu, \zeta)}{n-\zeta+2\nu} + \frac{S_b^2 b_2(\nu, \zeta)}{(n-\zeta+2\nu)(n-\zeta+2\nu-1)} + \dots \right), \quad (1.7)$$

where

$$\begin{aligned} A, \mathcal{M} = 2, \quad S_b = 2S_I = 2 \times 2 = 4 & \quad \text{DSG}, \\ A, \mathcal{M} = 1, \quad S_b = 2S_I = 2 \times \frac{1}{6} = \frac{1}{3} & \quad \text{TDW}. \end{aligned} \quad (1.8)$$

Here,  $S_I$  is the instanton action,  $S_b = 2S_I$  is the complex-bion action,  $A$  is the coefficient defined by (note that we set the natural frequency  $\omega$  to unity)

$$A = \lim_{t \rightarrow \pm\infty} \dot{x}_I(t) e^{|t|}, \quad (1.9)$$

where  $x_I(t)$  is the instanton solution, and  $\mathcal{M}$  comes from the multiplicity of the complex-bion solution<sup>6</sup>. At this stage,  $b_i(\nu, \zeta)$  just describe correction terms which can in general depend on  $\zeta$ .

The equation (1.7) is an insightful formula which deserves multiple comments. It indeed exhibits the generic  $\frac{n!}{(2S_I)^n}$  growth, but there is a curious  $\frac{1}{\Gamma(1+\nu-\zeta)}$  pre-factor which makes things rather interesting:

- 1a)** For  $\zeta \in \mathbb{N}^+$ , the leading asymptotic part of the perturbative expansion *vanishes* for level numbers  $\nu \leq |\zeta - 1|$ . By using an exact Bender-Wu analysis, we also demonstrate that the perturbation theory for those  $\zeta$  levels is *convergent*. The natural question is what is special for this class of theories?
- 1b)** For  $\zeta \notin \mathbb{N}^+$ , the perturbation theory for all levels is asymptotic. They are asymptotic in an expected manner  $\sim \frac{n!}{(2S_I)^n}$ . This is the generic behavior.

For  $\zeta = 1$ , the theory is supersymmetric and perturbation theory for the ground state  $\nu = 0$  is zero, but higher states show asymptotic expansions. For  $\zeta \in \mathbb{N}^+$  deformed theories, more states have convergent perturbation series. See Fig.1 which summarizes this perturbative findings.

### 1.3 The role of complex saddles in the semiclassical analysis

In Ref. [15], it was argued that consistent semi-classical analysis requires the inclusion of complex saddles in the semi-classical expansion. This requires the the real coordinate  $x(t)$  to be promoted to the complex coordinate  $z(t)$  and the path integral to be performed over complex integration cycles passing through the saddles. The saddles are, in general, solutions to the holomorphic Newton's

<sup>5</sup> We determined this growth in two independent ways. By using complex instanton/bion calculus, and assuming resurgence cancellation of the imaginary parts, this behavior is required. On the other hand we explicitly computed the perturbative coefficients via the **BenderWu** analysis, in excellent agreement with the instanton/bion calculus. The relation of the asymptotic growth to instantons and bions is the reason that the factors  $\mathcal{M}, A$  and  $S_b$ , characterizing the nature of the complex-bion appear. This will be discussed in detail in the rest of the paper.

<sup>6</sup>In other words, while there is only one complex solution contributing to the double well potential, for the double sine-Gordon there are two complex bion solutions.



equation in the inverted potential. In this way, a complex saddle solutions are found contributing to the ground state energy. Because of their complex nature and their relationship to instanton–anti-instanton, they are called complex bions<sup>7</sup>.

The leading non-perturbative contribution of complex bion  $[\mathcal{CB}]_{\pm}$  (or equivalently instanton–anti-instanton  $[\mathcal{II}]_{\pm}$ ) saddle to the energy level  $\nu$  is given by,

$$E_{\pm}^{\text{n.p.}}(\nu, g, \zeta) = [\mathcal{CB}]_{\pm} = [\mathcal{II}]_{\pm} = -\frac{1}{2\pi} \frac{\mathcal{M}}{\nu!} \left( \frac{g}{2A^2} \right)^{\zeta-2\nu-1} \Gamma(\zeta - \nu) e^{\pm i\pi(\zeta - \nu)} \times e^{-S_b/g} (b_0(\nu, \zeta) + b_1(\nu, \zeta) g + \dots), \quad (1.10)$$

where  $A$  and  $S_b$  are defined in (1.8). The exponent of  $e^{\pm i\pi(\zeta - \nu)}$  is the phase associated with the complex saddle and its descent manifold, and is called the *hidden topological angle (HTA)* [15, 30]. The sum  $\sum_{n \in \mathbb{N}} b_n(\nu, \zeta) g^n \equiv \mathcal{P}_{\text{fluc}}(\nu, g, \zeta)$  denotes perturbative fluctuations around the complex saddle contributions to level  $\nu$ . The HTA of the complex bion solution turns out to be extremely important for resolving some old standing puzzles stemming from the QES solutions.

The imaginary ambiguous parts of the complex bion amplitude can be found by using the reflection formula  $\Gamma(\zeta - \nu) \sin \pi(\zeta - \nu) = \frac{\pi}{\Gamma(1 - \zeta + \nu)}$  for the Gamma-function:

$$\text{Im } E_{\pm}^{\text{n.p.}}(\nu, g, \zeta) = \mp \frac{1}{2} \frac{\mathcal{M}}{\nu!} \left( \frac{g}{2A^2} \right)^{\zeta-2\nu-1} \frac{1}{\Gamma(1 + \nu - \zeta)} e^{-S_b/g} (b_0(\nu, \zeta) + b_1(\nu, \zeta) g + \dots) \quad (1.11)$$

Just like the Bender-Wu large order result (1.7), there is again intriguing structure associated with this formula which distinguishes  $\zeta \in \mathbb{N}^+$  due to the curious factor  $\frac{1}{\Gamma(1 + \nu - \zeta)}$  in (1.11):

- 2a)** For  $\zeta \in \mathbb{N}^+$ , for which hidden topological angle is quantized, the imaginary ambiguity in the energy disappears. What is again special for this class of theories?
- 2b)** For  $\zeta \notin \mathbb{N}^+$ , the Borel sum of (1.7) is ambiguous and has an imaginary ambiguity. This ambiguity must be exactly canceled by the imaginary part of the complex-bion contribution in (1.11), as the energy spectrum must be real.

We also note that there exists a real bion configuration for the DSG system, but there is no such configuration for the TDW. The real bion is a real saddle, and hence, it does not possess an HTA. The real bion contribution to energy level  $\nu$  is given by:

$$E^{\text{n.p.}}(\nu, g, \zeta) = [\mathcal{RB}] = [\mathcal{II}] = -\frac{\mathcal{M}}{2\pi} \frac{(-1)^\nu}{\nu!} \left( \frac{g}{2A^2} \right)^{\zeta-2\nu-1} \Gamma(\zeta - \nu) \times e^{-S_b/g} (b_0(\nu, \zeta) + b_1(\nu, \zeta) g + \dots) \quad (1.12)$$

Note that for the ground state is, the real bion always reduces the energy, while the for the higher states it alternate as  $(-1)^\nu$ . Also note that the multiplicity  $\mathcal{M}$  of the real bion is again  $\mathcal{M} = 2$ , just like that of the complex bion.

#### 1.4 Supersymmetry and Quasi-Exact Solvability

Both the quantization of the hidden topological angle as well as convergence of perturbation theory for  $\zeta \in \mathbb{N}^+$  suggest that there must be something very special about these QM systems. In particular,

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<sup>7</sup>The complex bion is an exact solution in the  $\zeta$ -deformed theory. It can equivalently be described as an instanton–anti-instanton correlated pair integrated over its quasi-zero mode Lefschetz thimble. We will use both instanton language and bion language interchangeably (see the discussion in [15]).

these systems must realize some generalization of the supersymmetric quantum mechanics. Indeed, this turns out to be the case.

For either  $W(x)$  given in (1.5) as well as a very large-class of  $W(x)$  studied in [55], we believe that perturbation theory for the lowest lying  $\zeta$  states is always convergent for  $\zeta \in \mathbb{N}^+$ . The question is whether there is a non-vanishing non-perturbative contribution or not? This is equivalent to the question of dynamical supersymmetry breaking in the  $\zeta = 1$  system:

- When  $e^{+W(x)}$  is normalizable, the first  $\zeta$  states of the  $\zeta$ -deformed theory are algebraically solvable with the following wave functions,

$$\Psi_i(x) = P_i(\xi(x))e^{+W(x)}, \quad i = 0, 1, \dots, \zeta - 1, \quad (1.13)$$

where  $P_i$  is a set of polynomials in the natural variable  $\xi(x)$  of the problem. For  $\zeta = 1$ , this means that supersymmetry is unbroken.

- When  $e^{+W(x)}$  is non-normalizable, non-vanishing non-perturbative contribution must exist. The reason is that this solution is generated by the perturbation theory, so it is an exact all-orders perturbative answer. However, since this state does not belong to the Hilbert space due to its non-normalizability, the true energy must be non-perturbatively shifted to amend it. The situation is entirely parallel in the case the supersymmetric limit when  $\zeta = 1$ , in which case the supersymmetry is dynamically broken by non-perturbative effects [47].

In both cases, the perturbation theory for the first  $\zeta \in \mathbb{N}^+$  states converges.

### 1.5 Two puzzles of QES

At this point, we wish to point out that our work also explains a puzzle emanating from the literature of the QES systems<sup>8</sup>:

**Puzzle 1)** For the DSG, the lowest  $\zeta$  states are algebraically solvable. The exact energy expressions which are algebraic in the coupling constant,  $g$ . At the same time, this system has obvious real instanton type saddles (what we called real bion). This would potentially give a non-algebraic contribution  $e^{-S_b/g}$  to the energy. From the exact solutions it can be explicitly seen that no such non-perturbative terms appear.

**Puzzle 2)** For the TDW, the lowest  $\zeta$  states are not algebraically solvable, these are not QES systems, but their perturbation theory converges. Since for the all-order perturbative result does not belong to the Hilbert space (i.e. is not normalizable), there must exist a non-perturbative shift in energy of the form  $e^{-S_b/g}$ , but there are no such real saddles for such a system.

We show that the resolution of these puzzles are given by the realization that apart from the real saddle contributions, there exists another contribution, the *complex bion*: **Puzzle 1** is solved because real and complex bions *exactly* cancel their non-perturbative contributions with each other for the lowest  $\zeta$  states; see Section 2. **Puzzle 2** is solved because there exist complex saddles contributing to the energy level  $\nu$ , and no real non-perturbative saddle to compensate it; see Section 3. This provides further evidence that complex paths and saddles are integral to the semi-classical expansion. Further we find that the convergence of the perturbation theory at these special points in the parameter space is *insufficient* in order to judge whether the perturbation theory gives an exact answer, and that

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<sup>8</sup>We would like to thank Edward Shuryak and Sasha Turbiner for drawing our attention to these puzzles.

cancellation between contributions of real and complex non-perturbative saddles gives a condition for the perturbation theory being exact. To employ the power of resurgence, such systems must be studied as integer limits of  $\zeta$ . This type of resurgence we call the Cheshire Cat resurgence, and discuss it next.



### 1.6 Cheshire Cat resurgence<sup>9</sup>

In our examples, the perturbation series is convergent for  $\zeta \in \mathbb{N}^+$ . As convergent series implies no ambiguity, it would seem that the role of resurgence in such systems is trivial as no cancellation between sectors is required, so we could not know whether complex bions contribute. We shall nevertheless see that convergence of the perturbation theory no longer holds under a tiny deformation of the theory, such as extension of  $\zeta \in \mathbb{N}^+$  to generic  $\zeta$ . Once this is done, the entire structure of resurgence is reestablished. All the relations obtained by resurgence survive even in the limit of convergent perturbation series. We will call it a *Cheshire Cat resurgence*, whose distinguishing features is that from time to time its body disappears, while its iconic grin remains.

For  $\zeta \notin \mathbb{N}^+$ , by using Bender-Wu analysis, we can do left/right resummation of perturbation theory, and prove that the ambiguity in the Borel sum  $\mathcal{S}_\pm E^{\text{pert.}}(\nu, g, \zeta)$  cancels exactly the ambiguity in the complex bion amplitude,  $[\mathcal{CB}]_\pm$ . Namely, at leading order, we obtain

$$\text{Im} \left[ \mathcal{S}_\pm E^{\text{pert.}}(\nu, g, \zeta) + [\mathcal{CB}]_\pm \right] = 0. \quad (1.14)$$

For  $\zeta \in \mathbb{N}^+$ , perturbation theory converges, and the complex bion amplitude becomes ambiguity free:

$$\text{Im} \mathcal{S}_\pm E^{\text{pert.}}(\nu, g, \zeta) = \text{Im} [\mathcal{CB}]_\pm = 0. \quad (1.15)$$

By using this relation, we find that the convergence of the perturbation theory corresponds to the quantization of the hidden topological angle to  $\theta_{\text{HTA}} \in \pi\mathbb{Z}$ . In order to extract the non-perturbative information from resurgence at  $\zeta \in \mathbb{N}^+$ , let us look more closely at their behaviors as a function of  $\zeta$ . For  $\zeta \rightarrow \mathbb{N}^+$ , we find that the imaginary part of the complex bion amplitude and the large-orders of perturbation theory behave as (for example, for ground state, setting  $\nu = 0$  in (1.11) and (1.7))

$$\begin{aligned} \text{Im} E_\pm^{\text{n.p.}}(\nu = 0, g, \zeta) &= \mp \frac{1}{2} \left( \frac{g}{2A^2} \right)^{\zeta-1} \frac{1}{\Gamma(1-\zeta)} e^{-S_b/g} (b_0(\nu, \zeta) + b_1(\nu, \zeta) g + b_2(\nu, \zeta) g^2 + \dots), \\ a_n(\nu = 0, \zeta) &= -\frac{\mathcal{M}}{2\pi} \frac{1}{(2A^2)^{\zeta-1}} \frac{1}{\Gamma(1-\zeta)} \frac{(n-\zeta)!}{(S_b)^{n-\zeta+1}} \\ &\quad \times \left( b_0(\nu, \zeta) + \frac{(S_b)b_1(\nu, \zeta)}{n-\zeta} + \frac{(S_b)^2 b_2(\nu, \zeta)}{(n-\zeta)(n-\zeta-1)} + \dots \right). \end{aligned} \quad (1.16)$$

Despite the fact that both expressions become zero in the  $\zeta \rightarrow \mathbb{N}^+$  limit due to overall  $\frac{1}{\Gamma(1-\zeta)}$  factor, and the resurgent cancellation seems to disappear, the footprint of resurgence is still present in the

<sup>9</sup>We are thankful to Roman Sulejmanpasic for the artwork.

theory. This may also be viewed as an analyticity in  $\zeta$ ; if resurgent cancellation works infinitesimal away from  $\zeta \in \mathbb{N}^+$ , its remnant must be present even in the limit.

By employing the Cheshire Cat resurgence we can justify our claim that the complex bion gives a contribution to the semiclassical analysis even when  $\zeta \in \mathbb{N}^+$ . This claim is the essential ingredient to solve the puzzles in QES literatures.

### 1.7 Self-resurgence and the Dunne-Ünsal relation

In this section we discuss another remarkable feature of the systems we study. Namely since in both systems we study the ambiguity of non-perturbative contribution is given by a complex saddle point—the complex bion in the cases we study—the perturbative corrections to this saddle will have a one-to-one correspondence with the corrections to the leading asymptotic growth of the perturbation theory. Although we always keep in mind the two systems we study (i.e. the DSG and the TDW), it is worth noting that the arguments we present in here are generally applicable to any system for which a complex saddle contributes to the energy shift, and for which the Dunne-Ünsal relation holds.

To show the self-resurgent properties, note that the coefficients  $b_i$  of the large order expansion (1.7) correspond to the perturbative corrections around the complex bion solution via the requirement of the imaginary part resurgent cancellation, i.e.

$$\text{Im } E_{\pm}^{\text{n.p.}}(\nu, g, \zeta) = (\dots) e^{-S_b/g} \text{Im}(e^{\pm i\zeta\pi}) \mathcal{P}_{\text{fluc}}(\nu, g, \zeta). \quad (1.17)$$

where  $\mathcal{P}_{\text{fluc}}(\nu, g, \zeta)$  is the perturbative expansion around the complex saddle, normalized so that  $\mathcal{P}_{\text{fluc}}(\nu, 0, \zeta) = 1$ . We write a formal expansion of this object as

$$\mathcal{P}_{\text{fluc}}(\nu, g, \zeta) = \sum_{i=0}^{\infty} b_i(\nu, \zeta) g^i, \quad (1.18)$$

where  $b_0 = 1$ . On the one hand the form of the large order growth (1.7) is fixed by the requirement that the ambiguity resulting from the complex bion is cancelled by the Borel summation of the perturbation theory. On the other hand, Dunne and one of us (MÜ) showed [9] that the perturbation around non-trivial saddles can be related to the perturbation theory around the trivial vacuum in a constructive way. By writing an analogous formula for the systems we study, we are able to relate the perturbative expansion around the trivial vacuum to the expansion around *the complex saddle*. These two facts then seem to imply that the perturbation theory around it both dictates and is dictated by (respectively) the late and early terms of the perturbation theory around the trivial vacuum. But if this is the case it means that the early terms of the perturbative series, “echoing” on the non-perturbative “mountain”, dictate late terms of *of the same series*. For this reason it is appropriate to call this phenomenon *echo-resurgence* or *self-resurgence*.

Let us see in more detail how this works. The formal power-expansion of the energy in coupling  $g$  of the energy level  $\nu$  is given by

$$E^{\text{pert.}}(\nu, g, \zeta) = a_0(\nu, \zeta) + a_1(\nu, \zeta)g + a_2(\nu, \zeta)g^2 + \dots \quad (1.19)$$

Now the natural generalization of the Dunne-Ünsal relation for an arbitrary complex bion is

$$\mathcal{P}_{\text{fluc}}(\nu, g, \zeta) = \frac{\partial E^{\text{pert.}}}{\partial \nu} \exp \left[ S_b \int_0^g \frac{dg}{g^2} \left( \frac{\partial E^{\text{pert.}}}{\partial \nu} - a'_0(\nu, \zeta) - a'_1(\nu, \zeta)g \right) \right]. \quad (1.20)$$

Writing

$$\frac{\partial E^{\text{pert.}}}{\partial \nu} = \sum_{n=0}^{\infty} a'_n(\nu, \zeta) g^n, \quad (1.21)$$

where the prime indicates differentiation with respect to  $\nu$ . We get that

$$\mathcal{P}_{\text{fluc}}(\nu, g, \zeta) = \left(1 + \sum_{n=1}^{\infty} a'_n(\nu, \zeta) g^n\right) e^{S_b \sum_{n=1}^{\infty} \frac{1}{n} a'_{n+1}(\nu, \zeta) g^n} \quad (1.22)$$

$$= (1 + a'_1(\nu, \zeta) g + a'_2(\nu, \zeta) g^2 + a'_3(\nu, \zeta) g^3 + \dots) \times \exp \left[ S_b \left( a'_2(\nu, \zeta) g + \frac{a'_3(\nu, \zeta) g^2}{2} + \frac{a'_4(\nu, \zeta) g^3}{3} + \dots \right) \right]. \quad (1.23)$$

Equivalently, we can write  $b_i(\nu, \zeta)$ 's in terms of derivatives of  $a_i(\nu, \zeta)$ 's,

$$\begin{aligned} b_0(\nu, \zeta) &= 1, \\ b_1(\nu, \zeta) &= a'_1(\nu, \zeta) + S_b a'_2(\nu, \zeta), \\ b_2(\nu, \zeta) &= a'_2(\nu, \zeta) + S_b a'_1(\nu, \zeta) a'_2(\nu, \zeta) + \frac{1}{2} S_b^2 a'_2(\nu, \zeta)^2 + \frac{1}{2} S_b a'_3(\nu, \zeta), \\ b_3(\nu, \zeta) &= a'_3(\nu, \zeta) + S_b \left( a'_2(\nu, \zeta)^2 + \frac{1}{2} a'_1(\nu, \zeta) a'_3(\nu, \zeta) + \frac{1}{3} a'_4(\nu, \zeta) \right) \\ &\quad + \frac{1}{2} S_b^2 (a'_1(\nu, \zeta) a'_2(\nu, \zeta)^2 + a'_2(\nu, \zeta) a'_3(\nu, \zeta)) + \frac{1}{6} S_b^3 a'_2(\nu, \zeta)^3. \end{aligned} \quad (1.24)$$

By plugging in (1.7) we get that the large order corrections of the perturbative expansion are given by

$$\begin{aligned} a_n(\nu, \zeta) &= -\frac{\mathcal{M}}{2\pi} \frac{1}{\nu!} \frac{1}{(2A^2)^{\zeta-2\nu-1}} \frac{1}{\Gamma(1+\nu-\zeta)} \frac{(n-\zeta+2\nu)!}{(S_b)^{n-\zeta+2\nu+1}} \\ &\times \left[ 1 + \frac{S_b (a'_1(\nu, \zeta) + S_b a'_2(\nu, \zeta))}{n-\zeta} + \right. \\ &\quad \left. + \frac{S_b^2 (a'_2(\nu, \zeta) + S_b a'_1(\nu, \zeta) a'_2(\nu, \zeta) + \frac{1}{2} S_b^2 a'_2(\nu, \zeta)^2 + \frac{1}{2} S_b a'_3(\nu, \zeta))}{(n-\zeta)(n-\zeta-1)} + \dots \right], \end{aligned} \quad (1.25)$$

$$(1.26)$$

what we have obtain is nothing short of remarkable! Indeed the expression above relates the asymptotic coefficients of the perturbation theory  $a_n$ , for  $n \gg 1$ , to the derivatives  $\partial_\nu a_n(\nu, \zeta) = a'_n(\nu, \zeta)$  for  $n = 1, 2, 3, \dots$ . For this reason we say that if the Dunne-Ünsal relation holds for a system in which complex saddles contribute, the perturbation series of the energy is said to be *self-resurgent*.

We take an opportunity now to comment on the possible interpretation of this self-resurgence formula as being related to Dingle's self-resurgence formula (see M.V. Berry [48]) which is a general property of resurgent functions which are themselves functions of resurgent functions. Namely it is likely<sup>10</sup> that the self-resurgent properties of the systems we study imply that the energy is not simply a resurgent function of two independent arguments  $\nu$  and  $g$ , but that they are related in some way. Indeed in [9], the energy is written as  $E(\nu(g), g)$ , where part of the dependence on  $g$  is placed into a functional dependence on  $\nu$ . On the other hand, here we obtained the self-resurgent formula by the utilization of the Dunne-Ünsal relation, a formula which is only known for systems who's WKB Riemann sheet is topologically a torus [23]. If by virtue of [48] the self-resurgent property is a general property of eigenvalue problems, this may give insight into what the generalization of the Dunne-Ünsal relation for higher genus WKB Reimann surfaces is.

<sup>10</sup>TS would like to thank M.V. Berry for drawing our attention to this possibility.

## 2 Resolving Puzzle 1: The Double Sine-Gordon system

In this section, we provide the resolution of the **Puzzle 1** that is described in Section 1.5. A subset of the lowest energy eigenstates of the Double Sine-Gordon (DSG) are exactly solvable, and the corresponding energy eigenvalues are known to be algebraic in coupling constant  $g$ . On the other hand, according to the textbook semi-classical analysis, the system possesses real instantons, what we call real bions. They should introduce non-algebraic  $e^{-S_b/g}$  contributions to the energy eigenvalues. The presence of complex bions, in addition to the real bions, lies in the heart of the solution to the apparent discrepancy. The complex bion contribution cancels precisely the one coming from real bions.

Here we consider the DSG system and analyze it in detail. The Hamiltonian is

$$H = -\frac{g}{2} \frac{\partial^2}{\partial x^2} + \frac{1}{2g} \left( W'(x)^2 - \zeta g W''(x) \right), \quad (2.1)$$

$$W(x) = -\omega \cos x,$$

$$V(x) = \frac{\omega^2}{2g} \sin^2 x - \zeta \frac{\omega}{2} \cos x, \quad (2.2)$$

where  $\omega$  is the curvature at  $x = 0$  to leading order in the expansion parameter  $g$  and  $\zeta$  is an a priori free parameter. The Schrödinger equation reads

$$-g \frac{\psi''(x)}{2} + V(x)\psi(x) = E\psi(x). \quad (2.3)$$

Since the potential is periodic, the wave-function can have Bloch periodicity  $\psi(x + 2\pi) = \psi(x)e^{i\theta}$ . By changing the  $\theta$ -angle we can scan the band of the potential. Below, we examine this class of potential by using the methods outlined in the Introduction for general values of  $\zeta$ .

First, recall that for the  $\zeta = 1$  case the above system reduces to the well known case of supersymmetric quantum mechanics, with the ground state energy  $E_0 = 0$  and the ground state wave-function

$$\psi_0 = e^{-W(x)/g} = e^{\frac{\omega}{g} \cos(x\sqrt{g})} \quad (2.4)$$

This solution determines the bottom of the lowest-lying band, or the ground state at  $\theta = 0$ . What is much less appreciated is that for any  $\zeta \in \mathbb{N}^+$ , it is always possible to find either the bottom or the top of the first  $\zeta$  bands (see Fig. 2) analytically. The method which allows one to determine these edges of the band goes under the name of *Quasi-Exact Solvability* (QES). We discuss this next.

### 2.1 Quasi-Exact Solvability for $\zeta \in \mathbb{N}^+$

The definition of the QES is that a finite part of the spectrum is algebraically exactly solvable (see [54] for a recent review of QES). Let us denote by  $\mathcal{H}_0$  the finite-dimensional subspace spanned by those eigenstates, which are algebraically solved under a certain boundary condition.

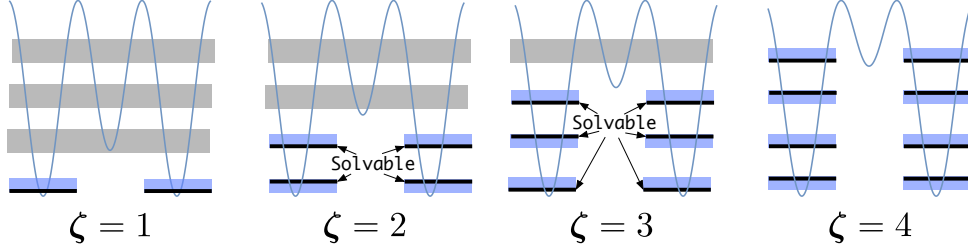
The method of QES relies on rewriting the eigenvalue problem by a suitably chosen Ansatz of the wave-function

$$\psi(x) = u(x)e^{-W(x)/g}. \quad (2.5)$$

The Schrödinger equation for  $\psi$  then turns into the eigenvalue equation for  $u(x)$ , of the form

$$\hat{h}u = Eu, \quad (2.6)$$

$$V(x) = \frac{\omega^2}{2g} \sin^2(x\sqrt{g}) - \zeta \frac{\omega}{2} \cos(x\sqrt{g})$$



**Figure 2.** An illustration of the exactly solvable states. The blue-shaded rectangles represent bands by changing the theta angle, whose width is non-perturbative and not exactly solvable for any  $\zeta$ . However, it is possible to solve either for the energy of the top or of the bottom of the band when  $\zeta$  is an odd or an even integer respectively. Note that  $\zeta = 1$  case is a supersymmetric limit, and the bottom of the band corresponds to the supersymmetric ground state given by  $\psi_0 = e^{-W(x)} = e^{\frac{\omega}{g} \cos(x\sqrt{g})}$ .

where  $\hat{h}$  is a second order differential operator given by

$$\hat{h} = \omega \left[ -\frac{g}{2\omega} \frac{d^2}{dx^2} + \sin x \frac{d}{dx} - \frac{\zeta - 1}{2} \cos x \right]. \quad (2.7)$$

Note that we can set  $\omega = 1$ . To reinstate it in the result we simply need to replace the coupling  $g \rightarrow g/\omega$  and the energy  $E \rightarrow \omega E$ .

In order to turn the Hamiltonian operator into a matrix eigenvalue equation in the subspace  $\mathcal{H}_0$ , it is useful to introduce differential operators which form a representation of  $SU(2)$  algebra,

$$J_+ = e^{-ix} \left( j - i \frac{d}{dx} \right), \quad J_- = e^{ix} \left( j + i \frac{d}{dx} \right), \quad J_3 = i \frac{d}{dx}, \quad (2.8)$$

with a Casimir  $(J_+ J_- + J_- J_+)/2 + J_3^2 = j(j+1)$ . The eigenfunctions of  $J_3$  are  $u_m = N e^{-imx}$ , with  $m = -j, -j+1, \dots, j$  and they form a multiplet in the  $2j+1$  dimensional representation of  $SU(2)$ .  $\mathcal{H}_0$  is the span of  $u_m$ , and exact solutions will be decomposable within this subspace.

The Hamiltonian  $\hat{h}$  given in (2.7) can be expressed in terms of generators (2.8)

$$\hat{h} = \frac{g}{2} J_3^2 - \frac{1}{2} (J_+ + J_-) = -\frac{g}{2} \frac{d^2}{dx^2} + \sin x \frac{d}{dx} - j \cos x, \quad (2.9)$$

provided we identify  $j = \frac{\zeta-1}{2}$ . Since  $j$  must be a non-negative integer or half-integer, we see that  $\zeta$  must be a positive integer.

Now note that there exists an abstract scalar product invariant under the action of the  $SU(2)$  group in question<sup>11</sup> and under which the states  $u_m$  are orthogonal, i.e.  $(u_m, u_s) = \delta_{ms}$  (see [59]). To

<sup>11</sup>Note that  $J_{\pm}$  are not Hermitian conjugates of each other under the naive  $L^2$  norm. We can, instead, introduce the invariant norm under the  $SU(2)$  group, so that it automatically makes the generators  $J_a$  invariant as it must follow that

$$(u, v) = (Uu, Uv) = (u, v) + t^a (iJ^a u, v) + t^a (u, iJ^a v) + o((t^a)^2) \Rightarrow (J^a u, v) = (u, J^a v),$$

where we wrote  $U = e^{iJ^a t^a}$ , where  $J^a$  are generators of  $SU(2)$ , and  $t^a$  are parameters of the transformation.

determine the norm of  $u_m$  we note that the action of  $J_{\pm}$  is given by

$$J_{\pm}u_m = \sqrt{(j \mp m)(j \pm m + 1)}u_{m \pm 1} . \quad (2.10)$$

Choosing  $u_{-j} \equiv e^{ijx}$ ,  $(u_{-j}, u_{-j}) \equiv 1$  by definition, we can construct all  $u_m$  by a successive action of  $J_+$ . This gives

$$u_m = \sqrt{\frac{(2j)!}{(j-m)!(j+m)!}} e^{-imx} , \quad m = -j, -j+1, \dots, j \quad (2.11)$$

This fact naturally splits the Hilbert space  $\mathcal{H}$  into a subspace invariant under the  $\hat{h}$  and the rest. Note that this  $SU(2)$  group is not a symmetry of the underlying theory, i.e. Hamiltonian is *not* invariant under the action of this  $SU(2)$ . This is clear from the form of the operator  $\hat{h}$ , given by (2.9), which is clearly not  $SU(2)$  invariant. Rather the space spanned by  $u_m$  is invariant under the action of  $\hat{h}$ , which allows for an algebraic solution of one part of the spectrum. Indeed in this subspace the operator  $\hat{h}$  attains a (tridiagonal) matrix form

$$\hat{h}_0 = \frac{1}{2} \begin{pmatrix} g(-j)^2 & -\sqrt{2j} & 0 & 0 & \dots & & & & \\ -\sqrt{2j} & g(-j+1)^2 & -\sqrt{2(2j-1)} & 0 & \dots & & & & \\ 0 & -\sqrt{2(2j-1)} & g(-j+2)^2 & \ddots & & & & & \\ 0 & 0 & \ddots & \ddots & & & & & \\ \vdots & \vdots & & & gm^2 & -\sqrt{(j-m)(j+m+1)} & & & \\ & & & -\sqrt{(j-m)(j+m+1)} & g(m+1)^2 & & & & \\ & & & & \ddots & \ddots & & & \\ & & & & & g(j-1)^2 & -\sqrt{2j} & & \\ & & & & & -\sqrt{2j} & gj^2 & & \end{pmatrix} , \quad (2.12)$$

where the subscript on  $\hat{h}$  implies the restriction of the total Hilbert space to the subspace  $\mathcal{H}_0$  invariant under the action of the  $SU(2)$  group generated by  $J_3, J_{\pm}$ . As we will only be concerned by this subspace, we will drop the subscript 0 in what follows.

### 2.1.1 Exact solutions for $\zeta = 1, 2, 3$ , and 4

Let us explicitly consider  $\mathcal{H}_0$  and  $\hat{h}$  for the cases  $\zeta = 1, 2, 3, 4$ .

#### $\zeta = 1$ (Supersymmetric) case: One exactly solvable state

This is the supersymmetric case, and it evidently requires  $j = 0$ , so that the only solvable state is  $u(x) = \text{const}$ . This is precisely the supersymmetric ground state. The Hamiltonian action on  $\mathcal{H}_0$  is

$$\hat{h} = 0, \quad (2.13)$$

with eigenvalues and eigenfunctions

$$E_0 = 0, \quad \psi_0 = e^{\frac{1}{g} \cos x}. \quad (2.14)$$

Notice that the wave-function is periodic, so it corresponds to the Bloch angle  $\theta = 0$ .



**$\zeta = 2$  case: Two exactly solvable states**

In this case  $j = 1/2$  and we have the Hamiltonian acting on  $\mathcal{H}_0$

$$\hat{h} = \begin{pmatrix} \frac{g}{8} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{g}{8} \end{pmatrix}, \quad (2.15)$$

with eigenvalues and eigenfunctions

$$\begin{aligned} E_0 &= -\frac{1}{2} + \frac{g}{8}; & \psi_0 &= \cos\left(\frac{x}{2}\right) e^{\frac{1}{g} \cos x}, \\ E_1 &= +\frac{1}{2} + \frac{g}{8}; & \psi_1 &= \sin\left(\frac{x}{2}\right) e^{\frac{1}{g} \cos x}. \end{aligned} \quad (2.16)$$

Notice that these wave-functions obey anti-periodic boundary condition, so that the Bloch angle is given by  $\theta = \pi$ .

**$\zeta = 3$  case: Three exactly solvable states**

If  $\zeta = 3$  then  $j = 1$ . The Hamiltonian is:

$$\hat{h} = \begin{pmatrix} \frac{g}{2} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{g}{2} \end{pmatrix}, \quad (2.17)$$

with eigenvalues and eigenfunctions

$$\begin{aligned} E_0 &= \frac{1}{4} \left( -g - \sqrt{g^2 + 16} \right); & \psi_0 &= \left( 2 \cos x + \frac{-g + \sqrt{g^2 + 16}}{2} \right) e^{\frac{1}{g} \cos x}, \\ E_1 &= \frac{g}{2}; & \psi_1 &= \left( \sin x \right) e^{\frac{1}{g} \cos x}, \\ E_2 &= \frac{1}{4} \left( -g + \sqrt{g^2 + 16} \right); & \psi_2 &= \left( 2 \cos x + \frac{-g - \sqrt{g^2 + 16}}{2} \right) e^{\frac{1}{g} \cos x}. \end{aligned} \quad (2.18)$$

Just like in the case when  $\zeta = 1$ , the all the wave-functions we found are periodic, therefore the Bloch angle is  $\theta = 0$  again.

**$\zeta = 4$  case: Four exactly solvable states**

If  $\zeta = 4$  then  $j = 3/2$ . The Hamiltonian is:

$$\hat{h} = \begin{pmatrix} \frac{9g}{8} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ -\frac{\sqrt{3}}{2} & \frac{g}{8} & -1 & 0 \\ 0 & -1 & \frac{g}{8} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{9g}{8} \end{pmatrix}, \quad (2.19)$$

with eigenvalues and eigenfunctions

$$\begin{aligned} E_0 &= -\frac{1}{2} + \frac{5}{8}g - \frac{\sqrt{4+2g+g^2}}{2}; & \psi_0 &= \left( \cos\left(\frac{3x}{2}\right) + \frac{\sqrt{g^2+2g+4+g+1}}{\sqrt{3}} \cos\left(\frac{x}{2}\right) \right) e^{\frac{1}{g} \cos x}, \\ E_1 &= +\frac{1}{2} + \frac{5}{8}g - \frac{\sqrt{4-2g+g^2}}{2}; & \psi_1 &= \left( \sin\left(\frac{3x}{2}\right) + \frac{\sqrt{g^2-2g+4+g-1}}{\sqrt{3}} \sin\left(\frac{x}{2}\right) \right) e^{\frac{1}{g} \cos x}, \\ E_2 &= -\frac{1}{2} + \frac{5}{8}g + \frac{\sqrt{4+2g+g^2}}{2}; & \psi_2 &= \left( \cos\left(\frac{3x}{2}\right) + \frac{-\sqrt{g^2+2g+4+g+1}}{\sqrt{3}} \cos\left(\frac{x}{2}\right) \right) e^{\frac{1}{g} \cos x}, \\ E_3 &= +\frac{1}{2} + \frac{5}{8}g + \frac{\sqrt{4-2g+g^2}}{2}; & \psi_3 &= \left( \sin\left(\frac{3x}{2}\right) + \frac{-\sqrt{g^2-2g+4+g-1}}{\sqrt{3}} \sin\left(\frac{x}{2}\right) \right) e^{\frac{1}{g} \cos x}. \end{aligned} \quad (2.20)$$

As in the case of  $\zeta = 2$ , wave-functions obey the anti-periodic boundary condition.

### 2.1.2 General $\zeta \in \mathbb{N}^+$ case and Ince-Polynomials

For general  $\zeta \in \mathbb{N}^+$  theory, the first  $\zeta$  level are algebraically solvable, corresponding to  $j = \frac{\zeta-1}{2}$  representation of  $SU(2)$ . The  $2j+1 = \zeta$  solutions are of the form

$$\psi_i(x) = P_i^{(\zeta-1)}(\cos(x/2), \sin(x/2)) e^{\frac{1}{g} \cos x}, \quad i = 0, 1, \dots, \zeta-1 \quad (2.21)$$

where  $P_i^{(\zeta-1)}(\cos(x/2), \sin(x/2))$  is an  $(\zeta-1)^{\text{th}}$  order polynomial with trigonometric arguments,  $\cos(x/2)$  and  $\sin(x/2)$ . These are called *Ince-polynomials* (see, e.g., Sec. 28.31 of Ref. [60]). The wave functions for the exactly soluble subset obey the boundary conditions:

$$\psi_i(x+2\pi) = (-1)^{\zeta-1} \psi_i(x) \quad (2.22)$$

- **$\zeta$ -odd:** In this case, the exactly solvable subset obey periodic boundary conditions (2.22). See Fig. 2. This corresponds to topological theta angle zero,  $\theta = 0$ . Note that for the  $\nu = 0$  band, this corresponds to the bottom of the band, while top of the band corresponds to  $\theta = \pi$ . The bottom of the  $\nu = 1$  band also corresponds to  $\theta = \pi$  and is again not algebraically solvable, but the top of the  $\nu = 1$  band correspond to  $\theta = 0$  and is algebraically solvable. This pattern continues for all odd- $\zeta$  values.
- **$\zeta$ -even:** In this case, the exactly solvable subset obey anti-periodic boundary conditions (2.22). See Fig. 2. This corresponds to setting topological theta angle to  $\theta = \pi$ . Note that for the  $\nu = 0$  band, this corresponds to the top of the band. Note that the bottom of the  $\nu = 1$  band also corresponds to  $\theta = \pi$  and it is also algebraically solvable. The rest of neither bands is exactly solvable. This pattern continues for all even- $\zeta$  values.

## 2.2 Complex saddles and the role of the hidden topological angle

There is a long-standing puzzle in the literature of the QES systems. Because of its algebraic nature, the exact solutions in QES systems have no non-perturbative contributions of the form  $e^{-S_b/g}$ . On the other hand, there is a real non-perturbative saddle in the DSG system, which we refer to as the real bion  $[\mathcal{RB}]$  [15]. This real saddle interpolates from  $x = 0$  to  $x = 2\pi$ , which may be interpreted as the exact version of instanton-instanton  $[ZZ]$  correlated event. There is no reason why this object would not contribute to the semiclassical analysis. Indeed such paths must be present, as only they couple to the Bloch  $\theta$  angle. If such paths did not contribute, the Bloch bands would not exist. Why does this contribution disappear in the exact energy expression? To our best knowledge, the resolution of this puzzle is not known in the QES literature.

The explanation of the above mentioned puzzle is very similar to that of the instanton-anti-instanton contributions in a supersymmetric theory [15]. In other words, for the case of  $\zeta = 1$  supersymmetric theory, the vacuum energy is zero to all orders in the perturbation theory. Contributions of the real bion renders the ground-state energy negative at  $\theta = 0$ , which clashes with the supersymmetry algebra. In [15], a complex multivalued saddle is found and called the complex bion, whose contribution to vacuum energy is positive,  $\Delta E^{cb} \sim -e^{\pm i\pi} e^{-S_b/g}$ , and it cancels the real-bion contribution  $\Delta E^{rb} \sim -e^{-S_b/g}$  exactly. This is the first hint for building up Picard-Lefschetz theory for path integrals<sup>12</sup>, because the complex saddles give must be included in the semiclassical expansion.

<sup>12</sup>As Picard-Lefschetz theory of path integrals is not a complete theory, we should clarify what we mean by the Picard-Lefschetz theory of path integrals. By Picard-Lefschetz theory we mean *a meaningful and systematic expansion of the observables which have a path-integral representation into contributions coming from various saddles of the action*. Note that we do not a priori refer to the nature, structure and construction of Lefschetz thimbles. More concretely, we do *not* claim that the dual of the Lefschetz thimble associated with the complex bion has nonzero intersection number with the original integration cycle.

In the  $\zeta$ -deformed theories, we find a similar exact cancellation mechanisms for  $\zeta \in \mathbb{N}^+$ .

- **$\zeta$ -odd:** As asserted above, the lowest  $\zeta$  states are exactly solvable at  $\theta = 0$ , and thus there must exist an exact cancellation between real and complex bion saddles. Indeed, we find

$$\begin{aligned} E^{\text{n.p.}}(\nu, g, \zeta) &= 2[\mathcal{RB}] + 2[\mathcal{CB}]_{\pm} \\ &= 2 \left( -(-1)^{\nu} - e^{\pm i\pi(\zeta - \nu)} \right)_{\zeta=1,3,\dots} e^{-S_b/g} \dots \\ &= 2(1 + e^{i\pi\zeta})_{\zeta=1,3,\dots} e^{-S_b/g} = 0 \end{aligned} \quad (2.23)$$

For level  $\nu = 0, 2, \dots, \zeta - 2$ , the real bion reduces the energy, while the complex bion increases it and the two cancel exactly. For level  $\nu = 1, 3, \dots, \zeta - 1$ , the real bion increases the energy, while the complex bion reduces it and the two cancel exactly. The cancellation between the two is a consequence of the destructive interference induced by *hidden topological angle*  $\theta_{\text{HTA}} = \zeta\pi$  associated with the complex saddle.

- **$\zeta$ -even:** In this case, the lowest  $\zeta$  states are exactly solvable at  $\theta = \pi$ , so we must consider the effect of the topological  $\theta$  angle for the QES. Since the real and complex bions have the winding numbers 1 and 0, respectively, we find that

$$[\mathcal{RB}](\theta) = [\mathcal{RB}]e^{i\theta}, \quad [\mathcal{CB}](\theta) = [\mathcal{CB}] \quad (2.24)$$

As a result of this, the contribution of real and complex bion to the energy level  $\nu$  for the case of even  $\zeta$  takes the form

$$\begin{aligned} E^{\text{n.p.}}(\nu, g, \zeta) &= ([\mathcal{RB}] + c.c.) + 2[\mathcal{CB}] \\ &= 2 \left( -(-1)^{\nu} \cos \theta|_{\theta=\pi} - e^{\pm i\pi(\zeta - \nu)} \right)_{\zeta=2,4,\dots} e^{-S_b/g} \dots \\ &= 2(\cos \theta|_{\theta=\pi} + e^{i\pi\zeta})_{\zeta=2,4,\dots} e^{-S_b/g} \dots = 0 \end{aligned} \quad (2.25)$$

For level  $\nu = 0, 2, \dots, \zeta - 1$ , the real bion increases the energy, while the complex bion reduces it and the two cancel exactly. For level  $\nu = 1, 3, \dots, \zeta - 2$ , the real bion reduces the energy, while the complex bion increases it and the two cancel exactly. The cancellation between the two is a consequence of the destructive interference induced also by *ordinary topological angle*  $\theta$  associated with the real saddle.

We find the mechanism described in (2.23) and (2.25) nothing short of remarkable. It is due to this exact non-perturbative cancellation mechanisms induced by the interplay of the hidden topological angle with the ordinary topological angle is necessary for the exact algebraic solvability of the states in  $\mathcal{H}_0$  in these QES-systems.

So far, we have shown that a consistent semiclassical picture is given for QES if we take into account the effect of complex bions. Remainder of Section 2 is dedicated to the analytic properties off the integer values of  $\zeta$ . As we shall see, such theories hold much more information about QES systems then would naively be thought.

### 2.3 The general $\zeta$ -deformed theory

In this and the next sections, we shall show that the complex bion must be taken into account for the semiclassical analysis by using a resurgence relation. For  $\zeta \in \mathbb{N}^+$  the perturbation series has a

finite convergence radius, and there seems to be no room for resurgence to play into the game. In order to understand what is happening better, we are going to compute the perturbation series of the DSG system for a generic  $\zeta \in \mathbb{R}$  and establish the intricate relation of the perturbative sector and the complex bion. By using the continuity in the limit  $\zeta \rightarrow 1, 2, 3, \dots$ , we argue that the complex bion still describes a nonperturbative contribution in the semiclassical analysis without the factorial growth of the perturbation series; we name it a Cheshire Cat effect.

To compute the perturbation series, we apply the Bender-Wu method [50, 51]. The Bender-Wu method is an algorithm to compute the high order correction for an arbitrary energy level in perturbation theory. The main idea of this algorithm is to construct the recursive relation for the perturbative coefficients of the eigen-energy  $E$  and eigenfunction  $\psi$ .

We demonstrate two remarkable aspect of the perturbation theory.

- For  $\zeta \in \mathbb{N}^+$ , the perturbation theory of the DSG system for  $\nu = 0, 1, \dots, \zeta - 1$  is convergent and exact. For higher energy levels, the perturbation theory yields an asymptotic expansion.
- For generic  $\zeta$ , the perturbation theory is always asymptotic.

### 2.3.1 Exactly solvability from perturbation theory for $\zeta \in \mathbb{N}^+$

For  $\zeta \in \mathbb{N}^+$ , Bender-Wu equation gives a convergent results for the energy levels  $\nu = 0, 1, 2, \dots, \zeta - 1$ , and it gives divergent asymptotic series for level number  $\nu \geq \zeta$ . See for example Tables 1a, 1b, 1c.

For  $\zeta = 1$ , the system is supersymmetric. Indeed, the ground state ( $\nu = 0$ ) energy is zero to all orders in perturbation theory. For the wave-function, perturbation theory does not yield zero, but a convergent and exact result for level number  $\nu = 0$ . For higher states  $\nu = 1, 2, \dots$  in the supersymmetric theory, perturbation theory is asymptotic.

As an example of a convergent (and non-truncating) perturbation theory, see Tables 1c, let us show the series for the the ground state energy of the  $\zeta = 3$  system.

$$E^{\text{pert.}}(\nu = 0, g, \zeta) = -1 + \frac{g}{4} - \frac{g^2}{32} + \frac{g^4}{2048} - \frac{g^6}{65536} + \frac{5g^8}{8388608} - \frac{7g^{10}}{268435456} + \dots, \quad (2.26)$$

which is exactly the expansion of  $E_0$  in (2.18).

### 2.3.2 Asymptotic corrections from the Bender-Wu analysis

Studying the Bender-Wu recursion relation, one can find the large-order behavior of perturbation theory.

$$a_n(\nu, \zeta) \approx -\frac{1}{\pi} \frac{1}{\nu!} \frac{1}{8\zeta-2\nu-1} \frac{1}{\Gamma(1+\nu-\zeta)} \frac{(n-\zeta+2\nu)!}{(S_b)^{n-\zeta+2\nu+1}} \times \left( 1 + \frac{S_b b_1(\nu, \zeta)}{n-\zeta+2\nu} + \frac{S_b^2 b_2(\nu, \zeta)}{(n-\zeta+2\nu)(n-\zeta+2\nu-1)} + \dots + \frac{S_b^K b_K(\nu, \zeta)}{(n-\zeta+2\nu)(n-\zeta+2\nu-1)\dots(n-\zeta+2\nu-K)} \right), \quad (2.27)$$

where we set  $b_0 = 1$  and terminated the  $1/n$  correction to some finite order  $K$ . This can be done for any state, but here we report for state  $\nu = 0$  and  $\nu = 1$ . We can then use the **BenderWu** package [49] to compute the coefficients  $a_n(\nu, \zeta)$  to some high order  $n = N$ , retaining the analytic dependence on  $\zeta$ . Explicit values of  $n = N - K, N - K + 1, \dots, N$  can then be plugged into the above approximate

equation, giving  $K$  equations with  $K$  unknowns  $b_1(\nu, \zeta), b_2(\nu, \zeta), \dots, b_K(\nu, \zeta)$ . Taking  $K = 10$ , and expanding in a series in  $\zeta$  for  $\nu = 0$  we get the following numerical values

$$\begin{aligned} b_1(\nu = 0, \zeta) &= -0.624999999802 + 0.624999999937376\zeta - 0.1249999999149091\zeta^2 + o(\zeta^3 10^{-11}), \\ b_2(\nu = 0, \zeta) &= -0.10156250436 + 0.015625013847\zeta + 0.117187481117609\zeta^2 \\ &\quad - 0.0624999852717885\zeta^3 + 0.00781249267805392\zeta^4 + o(\zeta^5 10^{-9}), \\ b_3(\nu = 0, \zeta) &= -0.116211 + 0.124022\zeta - 0.00325335\zeta^2 - 0.016603\zeta^3 - 0.003254481026038194\zeta^4 \\ &\quad + 0.002929444295569087\zeta^5 - 0.0003254657952886725\zeta^6 + o(\zeta^7 10^{-9}). \end{aligned} \quad (2.28)$$

We repeat the same for  $\nu = 1$  and obtain

$$\begin{aligned} b_1(\nu = 1, \zeta) &= -2.6249999967918 + 1.374999993327027\zeta - 0.1249999939430032\zeta^2 + o(\zeta^3 10^{-9}), \\ b_2(\nu = 1, \zeta) &= 1.24218678107 - 1.953123499260\zeta + 0.929686131557870\zeta^2 \\ &\quad + 0.1562492802762927\zeta^3 + 0.00781225722624248\zeta^4 + o(\zeta^5 10^{-8}), \\ b_3(\nu = 1, \zeta) &= -0.471608 + 0.473483\zeta - 0.444525\zeta^2 + 0.278248\zeta^3 - 0.0774493776897379\zeta^4 \\ &\quad + 0.00878340297785316\zeta^5 - 0.0003246026044364228\zeta^6 + o(\zeta^7 10^{-7}). \end{aligned} \quad (2.29)$$

The fact that the perturbative coefficients follow the factorial growth given by (1.7) suggests that the complex bion must contribute in the semiclassical analysis. By using the continuity in  $\zeta$ , complex bion gives the contribution also for  $\zeta \in \mathbb{N}^+$ , which solves the puzzle in QES literature as we discussed in Sec. 2.2. Furthermore, our detailed computation on  $b_i(\nu, \zeta)$  gives the conjecture about the perturbative fluctuations around the saddle of complex bions.

## 2.4 Self-resurgence and the Dunne-Ünsal relation

By using recursion relations, we can derive a perturbative expansion for the energy eigenvalues  $E^{\text{pert.}}(\nu, g, \zeta)$  as a function of coupling  $g$ , level number  $\nu$ , and parameter  $\zeta$ . For example, up to fourth order in  $g$ , we obtain an expression

$$E^{\text{pert.}}(\nu, g, \zeta) = a_0(\nu, \zeta) + a_1(\nu, \zeta)g + a_2(\nu, \zeta)g^2 + a_3(\nu, \zeta)g^3 + a_4(\nu, \zeta)g^4 + \mathcal{O}(g^5), \quad (2.30)$$

where

$$\begin{aligned} a_0(\nu, \zeta) &= \nu + \frac{1}{2} - \frac{\zeta}{2}, \\ a_1(\nu, \zeta) &= \frac{1}{8} (2\zeta\nu + \zeta - 2\nu^2 - 2\nu - 1) \\ a_2(\nu, \zeta) &= \frac{1}{64} \left( \zeta^2 (-(2\nu + 1)) + \zeta (6\nu^2 + 6\nu + 3) - 2 (2\nu^3 + 3\nu^2 + 3\nu + 1) \right), \\ a_3(\nu, \zeta) &= \frac{1}{256} \left( \zeta^3 (2\nu + 1) - 6\zeta^2 (2\nu^2 + 2\nu + 1) + \zeta (20\nu^3 + 30\nu^2 + 32\nu + 11) \right. \\ &\quad \left. - 2 (5\nu^4 + 10\nu^3 + 16\nu^2 + 11\nu + 3) \right), \\ a_4(\nu, \zeta) &= \frac{1}{4096} \left( -5\zeta^4 (2\nu + 1) + 48\zeta^3 (2\nu^2 + 2\nu + 1) - 2\zeta^2 (142\nu^3 + 213\nu^2 + 233\nu + 81) \right. \\ &\quad \left. + 15\zeta (22\nu^4 + 44\nu^3 + 74\nu^2 + 52\nu + 15) - 2 (66\nu^5 + 165\nu^4 + 370\nu^3 + 390\nu^2 + 225\nu + 53) \right) \end{aligned} \quad (2.31)$$

As stated earlier, the traditional resurgence connects large-order growth around the perturbative vacuum of perturbation theory to early terms around the instanton-anti-instanton saddle. However,

a new type of resurgence, which follows from exact quantization condition implemented via uniform WKB approach, offers a constructive version of resurgence. It is an early term–early term relation. The knowledge of perturbative expansion around the perturbative saddle at order  $g^n$  as a function of energy levels is sufficient to deduce the fluctuations around the the leading non-perturbative saddle at order  $g^{n-1}$ . The non-perturbative contribution to the energy for level  $\nu$  is

$$\begin{aligned} E_{\pm}^{\text{n.p.}}(\nu, g, \zeta) &= [\mathcal{RB}] + [\mathcal{CB}]_{\pm} \\ &= -\frac{1}{2\pi} \frac{2}{\nu!} \left(\frac{g}{8}\right)^{\zeta-2\nu-1} \Gamma(\zeta-\nu)((-1)^{\nu} + e^{\pm i\pi(\zeta-\nu)}) e^{-S_b/g} \mathcal{P}_{\text{fluc}}(\nu, g, \zeta) \end{aligned} \quad (2.32)$$

where  $\mathcal{P}_{\text{fluc}}(\nu, g, \zeta)$  is the fluctuation operator around the real and complex saddle. We remind the reader that, according to the result of [9, 10] (see also [61]) in the case of  $\zeta = 0$ , the fluctuations around an instanton-saddle are completely determined from the perturbative expansions around the trivial saddle. Inspired by this, we give a conjectured form of the relation between  $\mathcal{P}_{\text{fluc}}(\nu, g, \zeta)$  and the trivial perturbation theory  $E^{\text{pert.}}(\nu, g, \zeta)$

$$\mathcal{P}_{\text{fluc}}(\nu, g, \zeta) = \frac{\partial E^{\text{pert.}}}{\partial \nu} \exp \left[ S_b \int_0^g \frac{dg}{g^2} \left( \frac{\partial E^{\text{pert.}}}{\partial \nu} - 1 + \frac{2g(\nu + \frac{1}{2} - \frac{\zeta}{2})}{S_b} \right) \right]. \quad (2.33)$$

How can we check this formula? One way is to show consistency with the exact quantization condition, similar in spirit to the Zinn-Justin and Jentshura [56, 62, 63]. We defer the discussion of exact quantization condition for  $\zeta$ -deformed theories elsewhere. Instead, from above expression we can identify the  $\zeta$ -polynomials  $b_i(\nu, \zeta)$  by noting that

$$\mathcal{P}_{\text{fluc}}(\nu, g, \zeta) = b_0(\nu, \zeta) + b_1(\nu, \zeta)g + b_2(\nu, \zeta)g^2 + \dots, \quad (2.34)$$

so that

$$\begin{aligned} b_0(\nu, \zeta) &= 1, \\ b_1(\nu, \zeta) &= \frac{1}{8} \left( -\zeta^2 + \zeta(6\nu + 5) - 2\nu(3\nu + 5) - 5 \right), \\ b_2(\nu, \zeta) &= \frac{1}{128} \left( \zeta^4 - 4\zeta^3(3\nu + 2) + \zeta^2(48\nu^2 + 56\nu + 15) \right. \\ &\quad \left. - 2\zeta(36\nu^3 + 60\nu^2 + 30\nu - 1) + 36\nu^4 + 80\nu^3 + 60\nu^2 - 4\nu - 13 \right), \\ b_3(\nu, \zeta) &= \frac{1}{3072} \left( -\zeta^6 + 9\zeta^5(2\nu + 1) - 2\zeta^4(63\nu^2 + 51\nu + 5) + \zeta^3(432\nu^3 + 432\nu^2 + 42\nu - 51) \right. \\ &\quad \left. - 2\zeta^2(378\nu^4 + 444\nu^3 + 21\nu^2 - 165\nu + 5) + 3\zeta(216\nu^5 + 300\nu^4 - 228\nu^2 + 70\nu + 127) \right. \\ &\quad \left. - 3(72\nu^6 + 120\nu^5 - 152\nu^3 + 70\nu^2 + 254\nu + 119) \right). \end{aligned} \quad (2.35)$$

Setting  $\nu = 0$  and 1, we can compare them with an estimate to these polynomials in (2.28) and (2.29), respectively. Indeed the reader is welcome to check that the coefficients of (2.28) and (2.29) differ from the ones above by no more than 0.06%. This consistency again strengthens the evidence that the complex bion gives a physical contribution for the DSG system in the semiclassical analysis and

justifies the Cheshire Cat resurgence. For the ground state, set level number  $\nu = 0$  we obtain

$$\begin{aligned}
b_0(\nu = 0, \zeta) \Big|_{\text{D}\ddot{\text{U}}} &= 1, \\
b_1(\nu = 0, \zeta) \Big|_{\text{D}\ddot{\text{U}}} &= \frac{1}{8} \left( -5 + 5\zeta - \zeta^2 \right), \\
b_2(\nu = 0, \zeta) \Big|_{\text{D}\ddot{\text{U}}} &= \frac{1}{128} \left( -13 + 2\zeta + 15\zeta^2 - 8\zeta^3 + \zeta^4 \right), \\
b_3(\nu = 0, \zeta) \Big|_{\text{D}\ddot{\text{U}}} &= \frac{1}{3072} \left( -\zeta^6 + 9\zeta^5 - 10\zeta^4 - 51\zeta^3 - 10\zeta^2 + 381\zeta - 357 \right), \tag{2.36}
\end{aligned}$$

where we have explicitly indicated that the result was obtained from the Dunne-Ünsal relation (2.32). This confirms, at least to the precision indicated above that the formula (2.32) holds.

Let us do the same with  $\nu = 1$ . From the Dunne-Ünsal relation we have

$$\begin{aligned}
b_0(\nu = 1, \zeta) \Big|_{\text{D}\ddot{\text{U}}} &= 1, \\
b_1(\nu = 1, \zeta) \Big|_{\text{D}\ddot{\text{U}}} &= \frac{1}{8} \left( -\zeta^2 + 11\zeta - 21 \right), \\
b_2(\nu = 1, \zeta) \Big|_{\text{D}\ddot{\text{U}}} &= \frac{1}{128} \left( \zeta^4 - 20\zeta^3 + 119\zeta^2 - 250\zeta + 159 \right), \\
b_3(\nu = 1, \zeta) \Big|_{\text{D}\ddot{\text{U}}} &= \frac{1}{3072} \left( -\zeta^6 + 27\zeta^5 - 238\zeta^4 + 855\zeta^3 - 1366\zeta^2 + 1455\zeta - 1449 \right). \tag{2.37}
\end{aligned}$$

Comparing with (2.29) we find that the coefficients agree with the above formula to the precision of no more than 0.3% (most coefficients are below 0.06%).

### 3 Resolving Puzzle 2: Tilted Double-Well

In this section, we present the resolution of **Puzzle 2** of Section 1.5. The Tilted Double-Well (TDW) is not a QES system, but the perturbation series converge to an exact answer. This exact answer is non-normalizable, and therefore, it cannot be a non-perturbative solution. This is again in contrast with the existing textbook semi-classical approach since the TDW potential cannot exhibit real bion with finite energy. In the case of TDW, the complex bions come to rescue too and explain why perturbative solution is not exact.

Let us repeat our analysis for the  $\zeta$ -deformation of the double-well system to get more insight on the connection of the perturbation theory and complex saddles. The Hamiltonian takes the same form (1.4), where the auxiliary potential (or super-potential for  $\zeta = 1$ ) is given by

$$W(x) = \frac{x^3}{3} - \frac{\omega^2 x}{4}, \tag{3.1}$$

where  $\omega$  is the natural frequency of the system. For simplicity in the remainder of this section we set  $\omega = 1$ . We can always reinstate it by the following replacement  $x \rightarrow \omega x$ ,  $g \rightarrow g/\omega^3$  and the energy eigenvalues  $E \rightarrow \omega E$ .

Further the system also has a convergent perturbation series in powers of  $g$  for  $\zeta \in \mathbb{N}^+$ . Moreover, the series sums to a finite, but incorrect (or rather incomplete) result. We will derive this result analytically using techniques of QES. For generic  $\zeta$ , perturbation theory is asymptotic.

Note that subsection 3.1 should be considered as a review material as it is already discussed in literature in great depth [64, 65]. Here, we briefly discuss it for completeness. The relation of the fluctuations around the complex bion and perturbation theory around the trivial saddle (the Dunne-Ünsal relation) and the self-resurgence properties of the perturbation theory are new.

### 3.1 Pseudo-QES

We will now apply QES techniques to “solve” the TDW problem. We emphasize again, that this *not* a genuine solution to the full non-perturbative problem. It provides the all-order perturbative solution correctly but lacks some non-perturbative contributions.

We start, as usual, with an Ansatz

$$\psi(x) = u(x)e^{W(x)/g}, \quad (3.2)$$

motivated by the supersymmetric case  $\zeta = 1$ . This is non-normalizable solution to Schrödinger equation, but not a state in the Hilbert space. The equation for  $u(x)$  is given by

$$-\frac{g}{2}u''(x) - u'(x)W'(x) + \frac{1}{2}(\zeta - 1)u(x)W''(x) = Eu(x). \quad (3.3)$$

Plugging in  $W'(x) = x^2 - 1/4$ , we obtain the equation

$$-\frac{1}{2}g u''(x) + \left(\frac{1}{4} - x^2\right)u'(x) + x(\zeta - 1)u(x) = Eu(x). \quad (3.4)$$

Hence we define the reduced hamiltonian

$$\hat{h} = -\frac{g}{2}\frac{d^2}{dx^2} + \left(\frac{1}{4} - x^2\right)\frac{d}{dx} + x(\zeta - 1). \quad (3.5)$$

The objective now is to find eigenvalues  $E$  of this differential operator.

Now, observe that the operators

$$J_+ = 2jx - x^2\frac{d}{dx}, \quad J_- = \frac{d}{dx}, \quad J_3 = x\frac{d}{dx} - j. \quad (3.6)$$

obey the  $SU(2)$  algebra, which for  $2j \in \mathbb{N}^0$  leaves invariant the vector space spanned by polynomials  $u_m = N_m x^{j+m}$  for  $m = -j, -j+1, \dots, j$ . Further, it takes little to check that  $h_T$  can be written as

$$\hat{h} = -\frac{g}{2}J_-^2 + \left(\frac{1}{2}J_- + J_+\right) + (\zeta - 1 - 2j)x. \quad (3.7)$$

Since  $2j \in \mathbb{N}^0$ , then choosing  $\zeta = 2j + 1$  allows us to eliminate the last term above, and write  $h_T$  entirely in terms of operators  $J_\pm, J_3$ . The  $h_T$  operator acting on the  $SU(2)$  invariant subspace spanned by polynomials  $u_m$  is therefore given by

$$\hat{h} = -\frac{g}{2}J_-^2 + (J_-/4 + J_+), \quad 2j = \zeta - 1. \quad (3.8)$$

Let us now solve several specific cases.

#### 3.1.1 $\zeta = 1, 2, 3, 4$ perturbatively exact solutions

We consider  $\mathcal{H}_0$  and  $\widehat{h}_T$  for the cases  $\zeta = 1, 2, 3, 4$ .

##### $\zeta = 1$ case (SUSY)

If  $\zeta = 1$ , then  $j = 0$  and the only state in the invariant subspace is  $u_0 = \text{const}$ . The Hamiltonian action on  $\mathcal{H}_0$  is

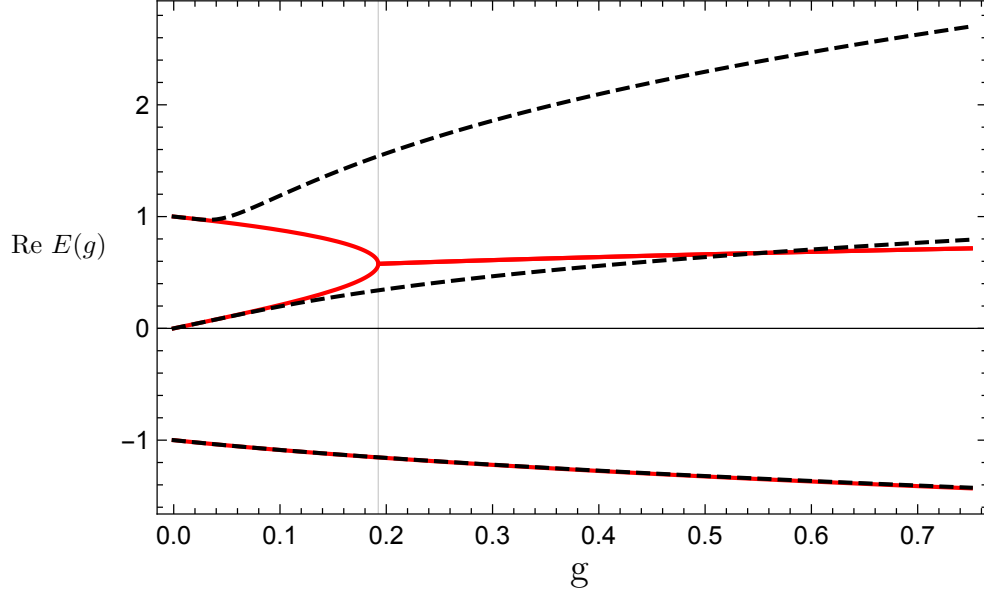
$$\hat{h} = 0 \quad (3.9)$$

with eigenvalues and eigenfunctions

$$E(\nu = 0) = 0, \quad (3.10)$$

This is indeed the result of the SUSY system, as  $e^{W(x)/g}$  solves the Schrödinger equation with energy zero. This state is not normalizable, hence, supersymmetry is broken dynamically. Indeed, a non-perturbative ground state energy has the form  $E_0^{\text{n.p.}} \sim e^{-S_b/g}$ .





**Figure 3.** A plot of eigenvalues  $E(\nu = \{0, 1, 2\}, g)$  for  $\zeta = 3$  as a function of coupling  $g$ . The solid lines represent the real part of the all orders in perturbation theory result (3.14), while the dashed lines represent the numerical solution to the Schrödinger equation. Notice that  $E(\nu = 1, g)$  and  $E(\nu = 2, g)$  in (3.14) collide and turn into complex conjugate pairs when  $g = \frac{1}{3\sqrt{3}}$ .

#### $\zeta = 2$ case

In this case we have to solve for eigenvalues of the matrix

$$\hat{h} = \begin{pmatrix} 0 & \frac{1}{4} \\ 1 & 0 \end{pmatrix} \quad (3.11)$$

which are simply

$$E(\nu = 0) = -\frac{1}{2}, \quad E(\nu = 1) = \frac{1}{2}. \quad (3.12)$$

#### $\zeta = 3$ case

Now the matrix becomes

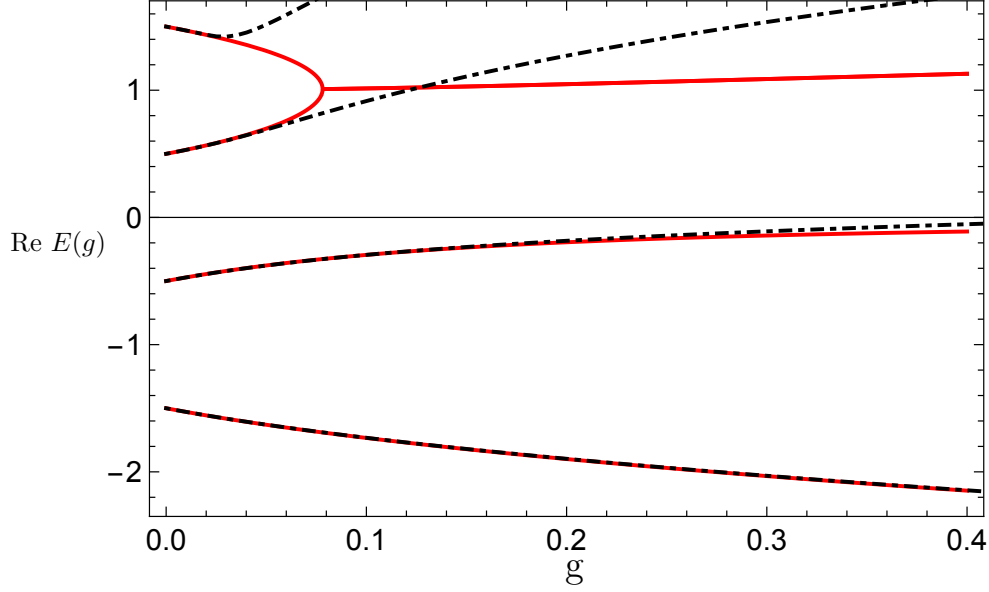
$$\hat{h} = \begin{pmatrix} 0 & \frac{1}{2\sqrt{2}} & -g \\ \sqrt{2} & 0 & \frac{1}{2\sqrt{2}} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \quad (3.13)$$

with eigenvalues

$$E(\nu = 0, g) = -\frac{2}{\sqrt{3}} \cos \left[ \frac{1}{3} \arccos [3\sqrt{3}g] \right] \quad (3.14a)$$

$$E(\nu = 1, g) = \frac{2}{\sqrt{3}} \sin \left[ \frac{1}{3} \arcsin [3\sqrt{3}g] \right] \quad (3.14b)$$

$$E(\nu = 2, g) = \frac{2}{\sqrt{3}} \cos \left[ \frac{1}{3} \arccos [-3\sqrt{3}g] \right] \quad (3.14c)$$



**Figure 4.** A plot of perturbative eigenvalues (solid line) and the numerical solutions (dashed line) for the states  $E(\nu = \{0, 1, 2, 3\}, g)$  for  $\zeta = 4$  as a function of coupling  $g$  which are solutions of (3.16). Notice that while  $E(\nu = \{0, 1\}, g)$  agree quite well (up to non-perturbative corrections), the perturbative values of  $E(\nu = 2, g)$  and  $E(\nu = 3, g)$  merge at some value of  $g$  and turn into complex conjugate pairs.

The plot of the real part of  $E(\nu = \{0, 1, 2\}, g)$  is given in Fig. 3, along with the numerical solution of the Schrödinger equation. Notice that while the ground state is described extremely well by the all-orders perturbative result<sup>13</sup>,  $E(\nu = 1, g)$  starts deviating significantly already at the coupling  $g \approx 0.1$ , while  $E(\nu = 2, g)$  shows a drastic deviation already at  $g \approx 0.05$ . Further when  $g = \frac{1}{3\sqrt{3}}$ , the two pseudo-eigenvalues  $E(\nu = 1, g)$ ,  $E(\nu = 2, g)$  merge and for  $g > \frac{1}{3\sqrt{3}}$  they become complex and turn into each other's complex conjugate pairs. This of course cannot happen for actual eigenvalues of the Schrödinger equation.

$\zeta = 4$  case

The  $\hat{h}$  matrix is given by

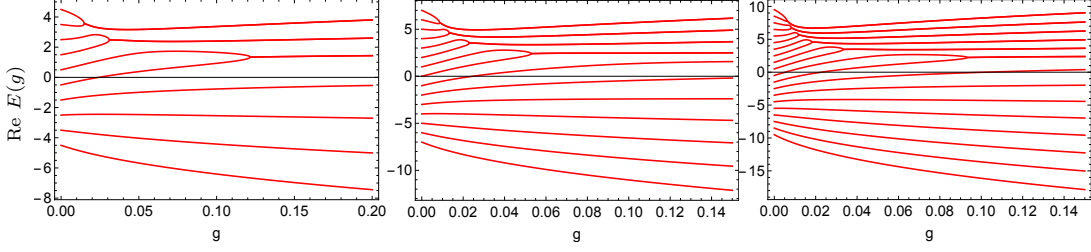
$$\hat{h} = \begin{pmatrix} 0 & \frac{\sqrt{3}}{4} & -\sqrt{3}g & 0 \\ \sqrt{3} & 0 & \frac{1}{2} & -\sqrt{3}g \\ 0 & 2 & 0 & \frac{\sqrt{3}}{4} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \quad (3.15)$$

with the characteristic equation

$$\left(E^2 - \frac{1}{4}\right) \left(E^2 - \frac{9}{4}\right) = -12gE \quad (3.16)$$

The form of the solutions is not particularly illuminating. We show in Fig. 4 a plot of the eigenvalues  $E(\nu = \{0, 1, 2, 3\}, g)$  which are the solution of the above equations, along with the numerical solution of the Schrödinger equation.

<sup>13</sup>It can be shown that the difference is in precise agreement with a complex bion for small  $g$ .



**Figure 5.** A plot of first  $\zeta$  eigenvalues to all order of perturbation theory for  $\zeta = 10, 15, 20$  (left to right).

### Large $\zeta$ case

We found it amusing to also discuss briefly a case where  $\zeta$  is large. Although solutions to the algebraic equation  $\hat{h}u = Eu$  do not have a nice closed form, they can nevertheless be easily plotted. In Fig. 5 we plot perturbative eigenvalues for three values of  $\zeta = 10, 15, 20$ . Notice that in all cases the top lying states merge into complex-conjugate pairs at some value of the coupling  $g$ . It is an interesting question of how and whether these complex parts can be cured by the non-perturbative contributions. Recall that the perturbation theory in all of these cases has a perfectly finite radius of convergence. Further these imaginary parts are completely unambiguous. We suspect that non-perturbative contributions must somehow contribute with an imaginary part, possibly as a result of a complete resummation of multi-instantons, in order to cancel this pathology. At this moment, however, this statement is speculative, and we leave it as an open problem.

### 3.2 Complex saddles and the role of the hidden topological angle

The non-normalizable states (3.2) provides a pseudo-QES system for which all orders perturbative results are obtained. But due to non-normalizability, these states are not a part of the Hilbert space, and all orders perturbative results cannot be correct expressions.

This is puzzling from a semi-classical point of view. Presumably, the all order perturbative result arises from the perturbative saddle, but there are no real non-perturbative saddles that can contribute to the path integral for the lowest  $\zeta$ -states. In the inverted potential,  $V(x) = W'(x)^2 + \zeta g W''(x)$ , a classical particle starting at the higher hill-top will over shoot the lower hill top and fly off to infinity. Thus, the action of such saddles is infinite and cannot contribute to semi-classical expansion of path integral. As discussed in depth in [15], the resolution of this puzzle is again given by complex bions.

The complex bion contribution to the energy for level  $\nu$  is given by (setting  $A = 1$  in (1.10), note that  $S_b = 1/3$ ), one finds

$$E_{\pm}^{\text{n.p.}}(\nu, g, \zeta) = [\mathcal{CB}]_{\pm} = -\frac{1}{2\pi} \frac{1}{\nu!} \left(\frac{g}{2}\right)^{\zeta-2\nu-1} \Gamma(\zeta - \nu) e^{\pm i\pi(\zeta-\nu)} e^{-S_b/g} (b_0(\nu, \zeta) + b_1(\nu, \zeta)g + \dots), \quad (3.17)$$

implying an imaginary ambiguous parts of the complex bion amplitude of the form,

$$\text{Im } E_{\pm}^{\text{n.p.}}(\nu, g, \zeta) = \mp \frac{1}{2} \frac{1}{\nu!} \left(\frac{g}{2}\right)^{\zeta-2\nu-1} \frac{1}{\Gamma(1 + \nu - \zeta)} e^{-S_b/g} (b_0(\nu, \zeta) + b_1(\nu, \zeta)g + \dots). \quad (3.18)$$

Few comments are in order:

- For generic  $\zeta$ , the contribution of the complex bion is two-fold ambiguous. This ambiguity cancels against the ambiguity in the Borel resummation of perturbation theory. If  $\zeta \in \mathbb{N}^+$ , the ambiguity vanishes for first  $\zeta$  states.

- **$\zeta$ -odd:** For level  $\nu = 0, 2, \dots, \zeta - 1$ , the complex bion increases the energy thanks to the hidden topological angle  $\theta_{\text{HTA}} = \pi$ . For levels  $\nu = 1, 3, \dots, \zeta - 2$ , the complex bion reduces the energy. The existence of these complex saddle gives non-perturbative contributions and is the reason that these states are not exactly solvable.
- **$\zeta$ -even:** For levels  $\nu = 0, 2, \dots, \zeta - 1$ , the the complex bion reduces the energy. For level  $\nu = 1, 3, \dots, \zeta - 2$ , the complex bion increases the energy compared to all order perturbative result.

Although we do not report here the details, all predictions arising from complex bions are realized in numerical solutions.

### 3.3 Bender-Wu method for $\zeta$ -deformed theory

The application of the Bender-Wu method to TDW system was performed using the **BenderWu** Mathematica package of [49]. Results are tabulated in Tables 2a, 2b, 2c for  $\zeta = 1, 2, 3$  for the lowest lying four levels  $\nu = 0, 1, 2, 3$  in Appendix B. Further using the **BenderWu** package we are able to construct the series as an analytic function of  $\zeta$ .

The main conclusion of these analysis are

- For positive integer  $\zeta$ , Bender-Wu approach yields a convergent perturbation theory for level number  $\nu = 0, 1, \dots, \zeta - 1$ . The summation of perturbation theory gives precisely the same result as in the pseudo-QES approach. These are all orders perturbative solutions to a non-perturbative problem. Unlike DSG, this is an incorrect, or rather incomplete, result. There exists non-perturbative corrections that arise from complex bion saddles. For higher states, Bender-Wu approach yields an asymptotic expansion.
- For generic  $\zeta$ , Bender-Wu approach yields an asymptotic perturbation theory, which can be viewed as the leading part of the resurgent trans-series.

#### 3.3.1 All orders perturbation theory

The TDW system with integer  $\zeta$  shows two types of behavior in perturbation theory. For the level numbers  $\nu = 0, 1, \dots, \zeta - 1$  the perturbation series is convergent, and it is asymptotic otherwise. Recall that we observed a similar behavior in the DSG system, where the perturbation theory summed to an exact result at an appropriate  $\theta$  angle. We will see, however, that the perturbation theory, although having a finite radius of convergence, gives an incorrect, or rather incomplete, result. In fact, we will show that the perturbation theory result is obtained exactly (i.e. to all orders) from an ansatz in the wave-function  $P(x)e^{W(x)/g}$ , where  $P(x)$  is a polynomial of order  $\zeta - 1$ . Such a result is clearly non-normalizable<sup>14</sup>, and is therefore inadmissible as a solution.

This is in contrast to the exactly solvable states of the DSG example in which the real bion cancels exactly the complex bion contribution, and convergent perturbation theory yields exact results. In the present case, there are no real saddle contribution to cancel the complex bion contribution. The non-perturbative contribution of the complex bion is, at leading order of the form,

$$\Delta E^{\text{n.p.}} = -\frac{g^{\zeta-1}}{2\zeta\pi}\Gamma(\zeta)e^{\pm i\pi\zeta}e^{-S_b/g} + \dots \quad (3.19)$$

---

<sup>14</sup>As we discussed, it is most convenient to build the perturbation theory in the canonical normalization, in which replaces  $x \rightarrow \sqrt{g}x$ . The wave-function  $\psi \propto e^{W(\sqrt{g}x)/g}$  is then easily seen to be normalizable to any finite order in perturbation theory by expanding it around the global minimum  $x = -a/\sqrt{g}$ . In other words the perturbation theory is oblivious to the global boundary conditions.

If  $\zeta$  is an odd integer,  $\zeta = 1, 3, 5, \dots$ , the contribution of the complex bion is positive. Note that  $\zeta = 1$  case is supersymmetric. For the present potential, supersymmetry is dynamically broken, and ground state energy is positive. In the bosonized language of the supersymmetric theory, the positivity of the ground state energy is due to the fact that the hidden topological angle is  $\theta_{\text{HTA}} = \pi$ .

For  $\zeta = 1$ , of course, perturbation theory is convergent and gives zero to all orders for the energy for the ground state  $\nu = 0$ . The perturbation theory for the wave function is convergent and upon summation, produces  $\psi \propto e^{W(\sqrt{g}x)/g}$ , the non-normalizable (perturbative) solution to the Schrödinger equation.

For  $\zeta = 2$  the perturbation theory is again rather trivial for the first two levels  $\nu = 0, 1$ , giving

$$E(\nu = 0) = -a, \quad E(\nu = 1) = +a, \quad a = \frac{1}{2}. \quad (3.20)$$

The reason for this is, as was pointed out in [66, 67], that the system can be related to a two supersymmetric system with the substitution  $\tilde{W}(x) = W(x) - \frac{1}{x \pm a}$ . The ground states of these SUSY systems correspond to the ground state and the first excited state<sup>15</sup> of the  $\zeta = 2$  tilted double well, and their perturbation theory is protected by supersymmetry.

For  $\zeta = 3$ , perturbation theory is convergent, but non-trivial. For the lowest lying three states, few terms in the perturbative expansion and their sum gives:

$$\begin{aligned} E^{\text{pert.}}(\nu = 0, g) &= -1 - g + \frac{3g^2}{2} - 4g^3 + \frac{105g^4}{8} - 48g^5 + \frac{3003g^6}{16} - 768g^7 + \frac{415701g^8}{128} - 14080g^9 + O(g^{10}) \\ &= -\frac{2}{\sqrt{3}} \cos\left(\frac{1}{3} \arccos(3\sqrt{3}g)\right), \\ E^{\text{pert.}}(\nu = 1, g) &= 0 + 2g + 8g^3 + 96g^5 + 1536g^7 + 28160g^9 + 559104g^{11} + O(g^{12}) \\ &= \frac{2}{\sqrt{3}} \sin\left(\frac{1}{3} \arcsin(3\sqrt{3}g)\right), \\ E^{\text{pert.}}(\nu = 2, g) &= +1 - g - \frac{3g^2}{2} - 4g^3 - \frac{105g^4}{8} - 48g^5 - \frac{3003g^6}{16} - 768g^7 - \frac{415701g^8}{128} - 14080g^9 - O(g^{10}) \\ &= \frac{2}{\sqrt{3}} \cos\left(\frac{1}{3} \arccos(-3\sqrt{3}g)\right). \end{aligned} \quad (3.21)$$

The perturbation theory has a finite radius of convergence for these three lowest lying states. The radius of convergence is

$$g \leq g_c = \frac{1}{3\sqrt{3}}. \quad (3.22)$$

Recall that  $g_c$  is a branch point of  $\arcsin(3\sqrt{3}g)$  and  $\arccos(3\sqrt{3}g)$ .

Note that for  $g < g_c$ , all these three solutions are real. At  $g = 0$  these solutions start at  $-1, 0, 1$ . Perturbative eigenvalue spectrum changes as a function of  $g$  for  $g < g_c$ , but at  $g = g_c$ , two real higher eigenvalues collide and move to the complex plane, with real and imaginary parts. This perturbative conclusion is obviously incorrect, but it is not currently clear what is the mechanism which turns these complex eigenvalues of the convergent perturbation theory and QES solution into real ones.

Unlike the text-book examples of saddles such as instantons, in which, instantons lead to level splitting of otherwise degenerate levels, in the present case, the complex bions lead to either up or down shift of the energy compared to all order perturbative result. It is still meaningful to include non-perturbative contribution, because all orders perturbative result is known exactly.

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<sup>15</sup>Note that the map  $\tilde{W}'(x) = \tilde{W}(x) - \frac{1}{x \pm a}$  is singular at either  $x = -a$  or at  $x = +a$ . Because of this the map disallows decomposition of the Hamiltonian into operators

### 3.3.2 Self-resurgence and the Dunne-Ünsal relation

Performing the Bender-Wu analysis via the **Mathematica** package **BenderWu**, we can find the sub-leading corrections to the leading factorial growth, for example, for the ground state energy we have:

$$a_n(\nu, \zeta) = \frac{1}{2\pi} \frac{1}{(2)^{\zeta-2\nu-1}} \frac{1}{\Gamma(1+\nu-\zeta)} \frac{(n+2\nu-\zeta)!}{(S_b)^{n-\zeta+1}} \times \left[ b_0(\nu, \zeta) + \frac{S_b b_1(\nu, \zeta)}{n-\zeta+2\nu} + \frac{S_b^2 b_2(\nu, \zeta)}{(n-\zeta+2\nu)(n-\zeta+2\nu-1)} + \dots \right]. \quad (3.23)$$

$b_0 = 1$  and where the pre-factor is constrained by demanding that the Borel sum ambiguity of the leading asymptotic growth of the perturbation series is exactly cancelled by the complex bion contribution to the ground state energy.

Further by using the **BenderWu** package [49] we can obtain a perturbative expansion for the energy eigenvalues as a function of coupling  $g$ , level number  $\nu$ , and parameter  $\zeta$ . For example, up to fourth order in  $g$ , we obtain an expression

$$E^{\text{pert}}(\nu, g, \zeta) = a_0(\nu, \zeta) + a_1(\nu, \zeta) g + a_2(\nu, \zeta) g^2 + a_3(\nu, \zeta) g^3 + a_4(\nu, \zeta) g^4 + \mathcal{O}(g^5), \quad (3.24)$$

where

$$\begin{aligned} a_0(\nu, \zeta) &= \nu + \frac{1}{2} - \frac{\zeta}{2}, \\ a_1(\nu, \zeta) &= \frac{1}{2} (-\zeta^2 + 6\zeta\nu + 3\zeta - 6\nu^2 - 6\nu - 2), \\ a_2(\nu, \zeta) &= \frac{1}{4} (4\zeta^3 - 21\zeta^2(2\nu+1) + \zeta(102\nu^2 + 102\nu + 35) - 2(34\nu^3 + 51\nu^2 + 35\nu + 9)) \\ a_3(\nu, \zeta) &= \frac{1}{4} (-16\zeta^4 + 123\zeta^3(2\nu+1) - 2\zeta^2(498\nu^2 + 498\nu + 173) + 3\zeta(500\nu^3 + 750\nu^2 + 528\nu + 139) \\ &\quad - 2(375\nu^4 + 750\nu^3 + 792\nu^2 + 417\nu + 89)), \\ a_4(\nu, \zeta) &= \frac{1}{16} (336\zeta^5 - 3453\zeta^4(2\nu+1) + 8\zeta^3(5010\nu^2 + 5010\nu + 1753) \\ &\quad - 6\zeta^2(16330\nu^3 + 24495\nu^2 + 17483\nu + 4659) \\ &\quad + \zeta(106890\nu^4 + 213780\nu^3 + 230550\nu^2 + 123660\nu + 27073) \\ &\quad - 2(21378\nu^5 + 53445\nu^4 + 76850\nu^3 + 61830\nu^2 + 27073\nu + 5013)). \end{aligned} \quad (3.25)$$

Using the Dunne-Ünsal relation, the knowledge of perturbative expansion around the perturbative saddle at order  $g^n$  is sufficient to deduce the fluctuations around the the leading non-perturbative saddle at order  $g^{n-1}$ . The leading non-perturbative saddle is the complex bion. The non-perturbative contribution to the energy for level  $\nu$  is given by

$$E_{\pm}^{\text{n.p.}}(\nu, g, \zeta) = [\mathcal{CB}]_{\pm} = -\frac{1}{2\pi} \frac{1}{\nu!} \left(\frac{g}{2}\right)^{\zeta-2\nu-1} \Gamma(\zeta-\nu) e^{\pm i\pi(\zeta-\nu)} e^{-S_b/g} \mathcal{P}_{\text{fluc}}(\nu, g, \zeta) \quad (3.26)$$

where  $\mathcal{P}_{\text{fluc}}(\nu, g, \zeta)$  is the fluctuation operator around the complex bion saddle. According to the result of [9, 10] (see also [61]),  $\mathcal{P}_{\text{fluc}}(\nu, g, \zeta)$  is completely dictated by  $E^{\text{pert.}}(\nu, g, \zeta)$  in a constructive way.

$$\begin{aligned} \mathcal{P}_{\text{fluc}}(\nu, g, \zeta) &= \frac{\partial E^{\text{pert.}}}{\partial \nu} \exp \left[ S_b \int_0^g \frac{dg}{g^2} \left( \frac{\partial E^{\text{pert.}}}{\partial \nu} - 1 + \frac{2g(\nu + \frac{1}{2} - \frac{\zeta}{2})}{S_b} \right) \right] \\ &= b_0(\nu, \zeta) + b_1(\nu, \zeta) g + b_2(\nu, \zeta) g^2 + b_3(\nu, \zeta) g^3 + \dots, \end{aligned} \quad (3.27)$$

where

$$\begin{aligned}
b_0(\nu, \zeta) &= 1, \\
b_1(\nu, \zeta) &= \frac{1}{6} (-21\zeta^2 + 3\zeta(34\nu + 23) - 6\nu(17\nu + 23) - 53), \\
b_2(\nu, \zeta) &= \frac{1}{72} \left( 441\zeta^4 - 36\zeta^3(119\nu + 60) + 3\zeta^2(4896\nu^2 + 4632\nu + 973) \right. \\
&\quad \left. - 6\zeta(3468\nu^3 + 4788\nu^2 + 1898\nu + 13) + 10404\nu^4 + 19152\nu^3 \right. \\
&\quad \left. + 11388\nu^2 + 156\nu - 1277 \right), \\
b_3(\nu, \zeta) &= \frac{1}{1296} \left( -9261\zeta^6 + 567\zeta^5(238\nu + 79) - 162\zeta^4(4879\nu^2 + 2883\nu + 143) \right. \\
&\quad \left. + 27\zeta^3(87856\nu^3 + 70704\nu^2 + 4014\nu - 4929) \right. \\
&\quad \left. - 18\zeta^2(213282\nu^4 + 214524\nu^3 + 7605\nu^2 - 45093\nu - 1465) \right. \\
&\quad \left. + 9\zeta(353736\nu^5 + 431460\nu^4 + 6336\nu^3 - 181836\nu^2 + 5794\nu + 45941) \right. \\
&\quad \left. - 1061208\nu^6 - 1553256\nu^5 - 28512\nu^4 + 1091016\nu^3 \right. \\
&\quad \left. - 52146\nu^2 - 826938\nu - 336437 \right). \tag{3.28}
\end{aligned}$$

On the other hand, we can find these coefficients  $b_1, b_2, b_3, \dots$  approximately from the explicit calculation of the perturbation theory. We get for  $\nu = 0$

$$\begin{aligned}
b_1(\nu = 0, \zeta) &= -8.83333332924 + 11.49999999835448\zeta - 3.499999997041193\zeta^2 + o(10^{-9}\zeta^3), \\
b_2(\nu = 0, \zeta) &= -17.736112193 - 1.0833289712\zeta + 40.5416588043985\zeta^2 \\
&\quad - 29.99999157670543\zeta^3 + 6.12499401506749\zeta^4 + o(10^{-6}\zeta^5), \\
b_3(\nu = 0, \zeta) &= -259.595 + 319.03\zeta + 20.3565\zeta^2 - 102.697\zeta^3 - 17.8679\zeta^4 + 34.559\zeta^5 \\
&\quad - 7.14458\zeta^6 + o(10^{-4}\zeta^7), \tag{3.29}
\end{aligned}$$

and for  $\nu = 1$

$$\begin{aligned}
b_1(\nu = 1, \zeta) &= -48.833333096294 + 28.49999939463613\zeta - 3.499999304322683\zeta^2 + o(10^{-7}\zeta^3), \\
b_2(\nu = 1, \zeta) &= 553.096586547 - 847.2483729877\zeta + 437.5397921020106\zeta^2 - 89.4987081484367\zeta^3 \\
&\quad + 6.12440133941877\zeta^4 + o(10^{-4}\zeta^5), \\
b_3(\nu = 1, \zeta) &= -2134.65 + 4591.34\zeta - 5398.5\zeta^2 + 3282.73\zeta^3 - 987.407646031719\zeta^4 \\
&\quad + 138.4467901243447\zeta^5 - 7.08375962476414\zeta^6 + o(10^{-2}\zeta^7). \tag{3.30}
\end{aligned}$$

The reader is welcome to check that (3.29) and (3.30) are numerically consistent with (3.28) at  $\nu = 0$  and  $\nu = 1$  within the relative error 0.05% and 0.8%, respectively.

## 4 Connection to Quantum Field Theory

Before conclusions we take an opportunity to comment on the potential significance of these systems and our motivation in studying them.

On the one hand, these quantum mechanical systems are helping us establish the rules of an all orders semi-classical expansion (i.e. exact semi-classics). On the other hand, these systems have

remarkable similarities with some quantum field theories, in particular, to gauge theories [57] and non-linear sigma models [19] with matter fields. In gauge theories, interactions between fermions and gauge fields are fixed by gauge invariance, and the  $\zeta$ -deformation of this interaction is forbidden. Adding a single adjoint Weyl fermion makes the gauge theory supersymmetric and corresponds to  $\zeta = 1$  QES theory. Higher integer  $\zeta$ -theories correspond to theories with higher number of flavors.

There is plentiful evidence that these theories are special. The first evidence of this is that the twisted (or graded) partition function (for SUSY it is simply the Witten index) defined by

$$Z = \text{tr} e^{-HL} (-1)^F, \quad (4.1)$$

where  $F$  is a fermion number, very likely has an absence of the confinement/deconfinement phase transition [57]. However, if the factor  $(-1)^F$  is dropped, the system has a thermal interpretation and undergoes a confinement/deconfinement transition at some  $L = L_c$  due to the Hagedorn growth of the density of states. This implies that  $F$ -even and  $F$ -odd cancel my cancel with an incredible (i.e. exponential) accuracy [58, 68, 69]. Further, in [30] it was shown that the non-perturbative contributions to the gluon condensate (and by trace anomaly to the vacuum energy) interfere with each other in these theories and their leading contribution vanishes. Further analogous multi-flavor  $CP^{N-1}$  systems show similar behavior [19].

Indeed, the direct analogue of these systems in Quantum Mechanics are precisely the integer  $\zeta$ -systems we study. These systems are special in that they have a fermionic symmetry and part of their spectrum is exactly solvable [55]. In this work, we understand the microscopic origins of these special features which we hope will help understand the analogous quantum field theories. While the situation in quantum field theories is undoubtedly much more subtle (i.e. except for supersymmetries, exact fermionic symmetries are forbidden in 4D by the Haag–Lopuszanski–Sohnius theorem) we cannot stay indifferent to the remarkable similarities between these two cases.

The general rule of obtaining such special theories is similar to promoting a theory to a supersymmetric one. A supersymmetric theory is obtained from a bosonic theory by an introduction of a single fermion flavor with the appropriate Lorentz and internal index structure. The special multi-flavor theories are obtained by simply adding more than one such flavor. This procedure is sketched in Quantum Mechanics as well as in the Yang-Mills and  $CP^{N-1}$  theories

	Bosonic	SUSY	multi-flavor generalization
<b>QM :</b>	$x(t)$	$\longrightarrow (x(t), \psi(t))$	$\longrightarrow (x(t), \psi^I(t))$ , $I = 1, \dots, n_f$
<b>Yang Mills :</b>	$A_\mu$	$\longrightarrow (A_\mu, \psi_\alpha)$	$\longrightarrow (A_\mu, \psi_\alpha^I)$ , $I = 1, \dots, n_f$
<b><math>CP^{N-1}</math> :</b>	$z_i$	$\longrightarrow (z_i, \psi_i)$	$\longrightarrow (z_i, \psi_i^I)$ , $I = 1, \dots, n_f$

## 5 Conclusion and Outlook

We study resurgence among the perturbation theory and non-perturbative complex saddles of the double sine-Gordon and tilted double-well quantum mechanics. These theories are obtained by deformation of the sine-Gordon and double-well quantum mechanics by introducing a parameter  $\zeta$ , respectively. Using the Bender-Wu method, we computed the perturbative coefficient of these theories as a function of  $\zeta$  and the level number  $\nu$ . By computing the perturbative coefficients explicitly we checked that the large-order asymptotic growth of the perturbation theory is correctly described by the complex-bion amplitude via traditional resurgence.



For both systems we study whenever  $\zeta \in \mathbb{N}^+$  the factorial growth of the perturbation theory vanishes for the first  $\zeta$  states. Using the technique of QES, we analytically show that this perturbation theory for the first  $\zeta$  states converges. This all-order perturbative solution gives an exact solution if it satisfies the correct boundary condition, but otherwise suffers from non-perturbative correction. There was a long-standing puzzle in the QES literature about this subtlety: the perturbative solution gives an exact solution while there exists a real non-perturbative classical solution, called a real bion, in one case, and the non-perturbative correction exists while a real bion is absent in the another case.

By analyticity in  $\zeta$  we conclude that the effect of complex saddle, called complex bion, is present for  $\zeta \in \mathbb{N}^+$  without any imaginary ambiguities, a phenomenon which we call the Cheshire Cat resurgence. We find that contributions of real and complex bions must be canceled in order for the convergent perturbative solution giving an exact answer. This emphasizes the importance of complex bion in the semiclassical analysis.

We also consider about the unconventional type of resurgent relation –the self-resurgence. In the double sine-Gordon and titled double-well cases, early terms of the expansion around the perturbative saddle give sufficient information about early terms of the expansion around complex bions. By exploiting the traditional resurgence, this means that early terms of the perturbative series know about late terms of the same series: i.e. the perturbative expansion is self-resurgent. We checked the self-resurgent property by explicitly computing the perturbative series, and found an astounding agreement.

It is an important future study to understand the effect of complex bions in the semiclassical analysis from the viewpoint of the path integral expression. Application of the Picard–Lefschetz theory to the (UV and IR regularized) path integral gathers much attention for numerical study of lattice field theories in order to tame the sign problem [70–73]. If the classical action takes complex values, then there exist situations where interference of multiple complex classical solutions are important for physical observables [74–84], which may remind us interference between real and complex bions. However, the models in this study do not have the sign problem since the classical action is a real functional at least when the coupling is physical<sup>16</sup>. This poses an interesting question on how we can understand the contribution of complex bions with nonzero HTA based on the Lefschetz-thimble decomposition of path integral.

Let us discuss some outlooks of this study toward the application of resurgence to certain quantum field theories. As we have already explained in the introduction, quantum mechanics studied in this paper has a formal similarity with adjoint QCD and some nonlinear sigma models, which possess some interesting properties as supersymmetric field theories. By comparing properties of the hidden topological angle, we can speculate that magnetic and neutral bions in adjoint QCD correspond to real and complex bions in quantum mechanics. It is an interesting topic to deduce nonperturbative criteria on phases of gauge theories by discussing constructive or destructive interference of real and complex bions [85]. It is also a great task to explain the relation between asymptotic nature of the perturbation theory and those bion solutions in quantum field theories [14, 21, 86].

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<sup>16</sup>A sign problem does exist in the formulation of (1.2), as the “Dirac operator” determinant is not positive definite. The study of this systems however can be reduced to the study of the  $\zeta$ -deformed systems with  $\zeta = -n_f/2, \dots, n_f/2$ , all of which do not possess the sign problem.

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## A Tables for perturbative coefficients

We here show tables of perturbative coefficients of the DSG and TDW systems at  $\zeta = 1, 2$ , and  $3$ .

	$\nu = 0$	$\nu = 1$	$\nu = 2$	$\nu = 3$
1	0	1	2	3
$g$	0	$-\frac{1}{4}$	$-1$	$-\frac{9}{4}$
$g^2$	0	$-\frac{3}{32}$	$-\frac{9}{16}$	$-\frac{57}{32}$
$g^3$	0	$-\frac{3}{32}$	$-\frac{27}{32}$	$-\frac{117}{32}$
$g^4$	0	$-\frac{273}{2048}$	$-\frac{1809}{1024}$	$-\frac{21027}{2048}$
$g^5$	0	$-\frac{15}{64}$	$-\frac{2295}{512}$	$-\frac{2205}{64}$
$g^6$	0	$-\frac{31443}{65536}$	$-\frac{425169}{32768}$	$-\frac{8546937}{65536}$
$g^7$	0	$-\frac{18141}{16384}$	$-\frac{339957}{8192}$	$-\frac{8846739}{16384}$
$g^8$	0	$-\frac{23700837}{8388608}$	$-\frac{603458181}{4194304}$	$-\frac{20134192143}{8388608}$
$g^9$	0	$-\frac{4126773}{524288}$	$-\frac{70008543}{131072}$	$-\frac{5933126187}{524288}$
$g^{10}$	0	$-\frac{6370376181}{268435456}$	$-\frac{282618414063}{134217728}$	$-\frac{15066484101519}{268435456}$

(a) DSG  $\zeta = 1$

	$\nu = 0$	$\nu = 1$	$\nu = 2$	$\nu = 3$
1	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$g$	$\frac{1}{8}$	$\frac{1}{8}$	$-\frac{3}{8}$	$-\frac{11}{8}$
$g^2$	0	0	$-\frac{3}{16}$	$-\frac{15}{16}$
$g^3$	0	0	$-\frac{15}{64}$	$-\frac{105}{64}$
$g^4$	0	0	$-\frac{417}{1024}$	$-\frac{4065}{1024}$
$g^5$	0	0	$-\frac{3525}{4096}$	$-\frac{47355}{4096}$
$g^6$	0	0	$-\frac{68433}{32768}$	$-\frac{1251345}{32768}$
$g^7$	0	0	$-\frac{739695}{131072}$	$-\frac{18168465}{131072}$
$g^8$	0	0	$-\frac{69923685}{4194304}$	$-\frac{2275665765}{4194304}$
$g^9$	0	0	$-\frac{892497165}{16777216}$	$-\frac{37956199155}{16777216}$
$g^{10}$	0	0	$-\frac{24410041071}{134217728}$	$-\frac{1337756678895}{134217728}$

(b) DSG  $\zeta = 2$

	$\nu = 0$	$\nu = 1$	$\nu = 2$	$\nu = 3$
1	$-1$	0	1	2
$g$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	$-\frac{1}{2}$
$g^2$	$-\frac{1}{32}$	0	$\frac{1}{32}$	$-\frac{5}{16}$
$g^3$	0	0	0	$-\frac{15}{32}$
$g^4$	$\frac{1}{2048}$	0	$-\frac{1}{2048}$	$-\frac{985}{1024}$
$g^5$	0	0	0	$-\frac{1215}{512}$
$g^6$	$-\frac{1}{65536}$	0	$\frac{1}{65536}$	$-\frac{218165}{32768}$
$g^7$	0	0	0	$-\frac{168945}{8192}$
$g^8$	$\frac{5}{8388608}$	0	$-\frac{5}{8388608}$	$-\frac{290599445}{4194304}$
$g^9$	0	0	0	$-\frac{32709615}{131072}$
$g^{10}$	$-\frac{7}{268435456}$	0	$\frac{7}{268435456}$	$-\frac{128334137795}{134217728}$

(c) DSG  $\zeta = 3$

**Table 1.** Tables of perturbative coefficients of the DSG system at  $\zeta = 1, 2$ , and  $3$ .

$\nu = 0$		$\nu = 1$		$\nu = 2$	
1	0	1		2	
$g$	0	-3		-12	
$g^2$	0	$-\frac{39}{2}$		-141	
$g^3$	0	-270		-3330	
$g^4$	0	$-\frac{41433}{8}$		$-\frac{418953}{4}$	
$g^5$	0	-121104		-3895866	
$g^6$	0	$-\frac{52149999}{16}$		$-\frac{1300600941}{8}$	
$g^7$	0	-97888095		-7397575110	
$g^8$	0	$-\frac{412171252725}{128}$		$-\frac{23088242197365}{64}$	
$g^9$	0	$-\frac{229284886527}{2}$		-18643301573274	
$g^{10}$	0	$-\frac{1121697677785665}{256}$		$-\frac{129681560992818075}{128}$	

(a) TDW  $\zeta = 1$

$\nu = 0$	$\nu = 1$	$\nu = 2$	$\nu = 0$	$\nu = 1$	$\nu = 2$	$\nu = 3$	
1	$-\frac{1}{2}$	$\frac{1}{2}$	1	-1	0	1	2
$g$	0	0	$g$	-1	2	-1	-10
$g^2$	0	0	$g^2$	$\frac{3}{2}$	0	$-\frac{3}{2}$	-105
$g^3$	0	0	$g^3$	-4	8	-4	-2290
$g^4$	0	0	$g^4$	$\frac{105}{8}$	0	$-\frac{105}{8}$	$-\frac{270705}{4}$
$g^5$	0	0	$g^5$	-48	96	-48	-2388810
$g^6$	0	0	$g^6$	$\frac{3003}{16}$	0	$-\frac{3003}{16}$	$-\frac{761448465}{8}$
$g^7$	0	0	$g^7$	-768	1536	-768	-4152537390
$g^8$	0	0	$g^8$	$\frac{415701}{128}$	0	$-\frac{415701}{128}$	$-\frac{12464873944005}{64}$
$g^9$	0	0	$g^9$	-14080	28160	-14080	-9704850553210
$g^{10}$	0	0	$g^{10}$	$\frac{15935205}{256}$	0	$-\frac{15935205}{256}$	$-\frac{65234638372161615}{128}$

(b) TDW  $\zeta = 2$

(c) TDW  $\zeta = 3$

**Table 2.** Tables of perturbative coefficients of the TDW system at  $\zeta = 1, 2$ , and 3.

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